Dynamics of Staircase Formation

Reduced Model of Beta Plane Turbulence

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Abstract / Introduction/Objectives

Abstract A two-field model for staircase dynamics relevant to both beta-plane geostrophic and drift-wave plasma turbulence is studied numerically and analytically. The model evolves an averaged potential vorticity (PV) whose flux is both driven by, and regulates, an enstrophy field, ε. The model's closure uses a mixing length concept. Its link with bistability, vital to staircase generation, is analysed and verified by integrating the equations numerically.

Introduction The turbulent transport and structure formation phenomenon known as a 'staircase', originally introduced in [2] manifests itself as follows:

- stably stratified density profile in the ocean occasionally reorganizes into layers separated by thin interfaces
- density gradient flattens in the layers and steepens in the interfaces →'staircase'
- pre-existing turbulent transport is supported by, and regulates, the gradient
- positive feedback provided by a profile rippling instability is equivalent to a 'negative diffusivity' that enhances the profile corrugation instead of smoothing it
- negative diffusion corresponds to a descending branch of an "S-curve" in the flux - gradient relation, i.e. a range of ∇n for which $\delta\Gamma/\delta\left(-\nabla n\right)<0$
- feedback loop drives the transport supporting turbulence out of the regions with steeper profiles into adjacent regions with the flatter ones, thus settling at a *bistable* equilibrium

Objectives

1. identification of conditions and the parameter space for staircase formation
2. demonstration of staircase persistence by direct numerical integration of the model equations
3. finding exact analytic steady state solutions and exploiting them for code verification
4. elucidation of staircase dynamics, long time evolution, merger events and the role of domain boundaries

Model:

$$Q_t = \partial_y \frac{\varepsilon^{1/2}}{\left(1+Q_y^2/\varepsilon\right)^2} Q_y + D Q_{yy} \qquad \varepsilon_t = \partial_y \frac{\varepsilon^{1/2}}{\left(1+Q_y^2/\varepsilon\right)^2}\varepsilon_y + D\varepsilon_{yy} + \frac{\varepsilon^{1/2}}{\left(1+Q_y^2/\varepsilon\right)^2} Q_y^2 - \frac{\varepsilon^{3/2}}{\varepsilon_0} + \gamma\sqrt{\varepsilon}$$

Formulation Consider potential vorticity (PV), q, of a geostrophic fluid, e.g., on a rapidly rotating planet. It consists of the planetary vorticity (on β-plane) and fluid vorticity:

$$q = \beta y + \Delta\psi$$

where ψ is the stream function, and y is a latitudinal coordinate. Equation for q:

$$\frac{\partial q}{\partial t} - \nabla\psi\times\nabla q = \nu\Delta\psi + f \qquad (1)$$

Decompose q and ψ into a mean and fluctuating parts

$$q = \langle q\left(y,t\right)\rangle + \tilde{q}\left(x,y,t\right)$$

with $\tilde{q}=\Delta\tilde{\psi}$. Separate the x-averaged component $Q\equiv\langle q\rangle$ from fluctuating part squared (enstrophy), $\varepsilon=\left\langle\tilde{q}^2\right\rangle/2$. The closure problem for $\langle\nabla\tilde{\psi}\times\nabla\Delta\tilde{\psi}\rangle$ arises. For fluctuations statistically homogeneous in x-direction the x-averaged PV flux Γ_q is:

$$-\frac{\partial\Gamma_q}{\partial y} \equiv \langle\nabla\tilde{\psi}\times\nabla\Delta\tilde{\psi}\rangle = \frac{\partial^2}{\partial y^2}\left\langle\frac{\partial\tilde{\psi}}{\partial x}\frac{\partial\tilde{\psi}}{\partial y}\right\rangle.$$

Next, we apply a Fickian *Ansatz*: $\Gamma_q = -D_e\partial Q/\partial y$, where $D_e\left(\varepsilon,Q_y\right)$ is the PV diffusivity. This is assumed to follow a mixing-length hypothesis, $D_e\sim l\left|\nabla\tilde{\psi}\right|$, where $l\left(\varepsilon,Q_y\right)$ is the mixing length, introduced phenomenologically as [1]:

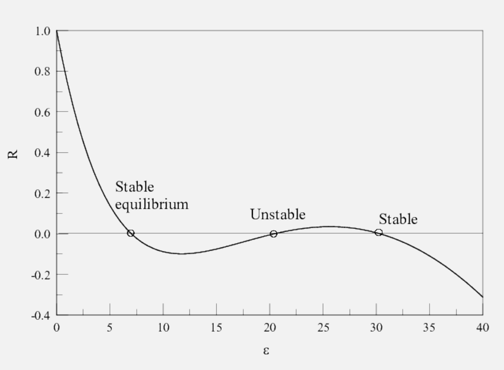
$$\frac{1}{l^2}=\frac{1}{l_0^2}+\frac{1}{l_R^2}. \qquad (2)$$

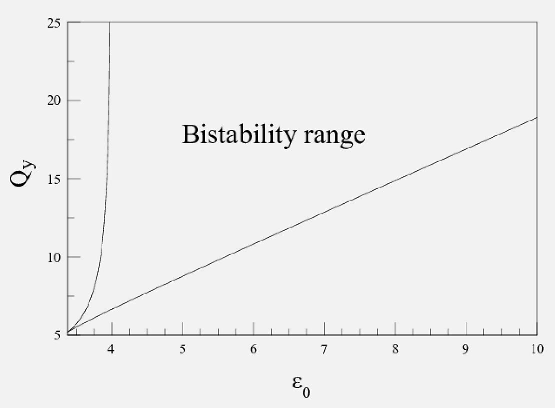
Here, l_0 is a fixed contribution to the mixing length l that characterizes the turbulence, e.g., the stirring scale. l_R is the Rhines scale at which dissipation of ε balances its production, so $l_R = l_R\left(\varepsilon,Q_y\right)$. In turbulent cascades where wave form of energy coexists with turbulent eddies, the Rhines scale is where these two intersect, i.e., where $k\tilde{v}\sim\omega_k$ [3]. When the turbulent energy inverse cascade reaches this scale, it is intercepted and transported further by waves both in wave-number and configuration space. The only dimensionless combination of the variables entering eq.(2) is $l_0^2Q_y^2/\varepsilon$. So, we may generalize the relation in eq.(2) and write $l_0/l=\left(1+l_0^2Q_y^2/\varepsilon\right)^{\kappa}$. We choose $\kappa=2$. Replacing the eddy velocity in the Fick's law by $l_0\sqrt{\varepsilon}$ and measuring y in units of l_0, we can write the averaged eq.(1) for Q as shown above with added (small) constant diffusivity D. Applying similar arguments to the turbulent part of PV, ϵ, and adding the terms responsible for its production, damping and unstable growth, we obtain the above evolution equation for the enstrophy ε.

Staircase Prerequisites/Formation/Merger

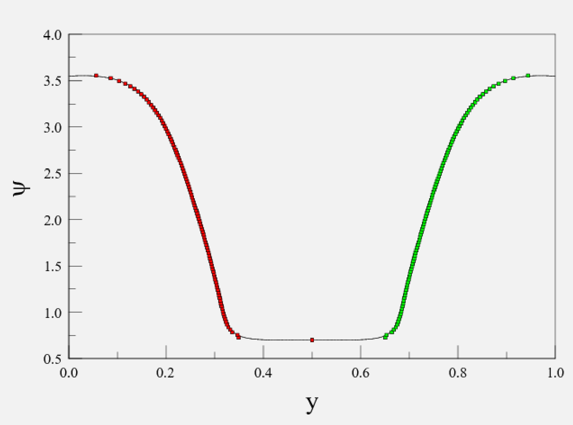
- SC result from the loss of stability of a ground state solution for Q and ϵ, characterized by the constant values $\varepsilon=\varepsilon_B$ and $Q_y=Q_B'$ that annihilate the enstrophy production-dissipation term:

$$R\equiv\frac{Q_y^2}{\left(1+Q_y^2/\varepsilon\right)^2}-\frac{\varepsilon}{\varepsilon_0}+\gamma=0$$



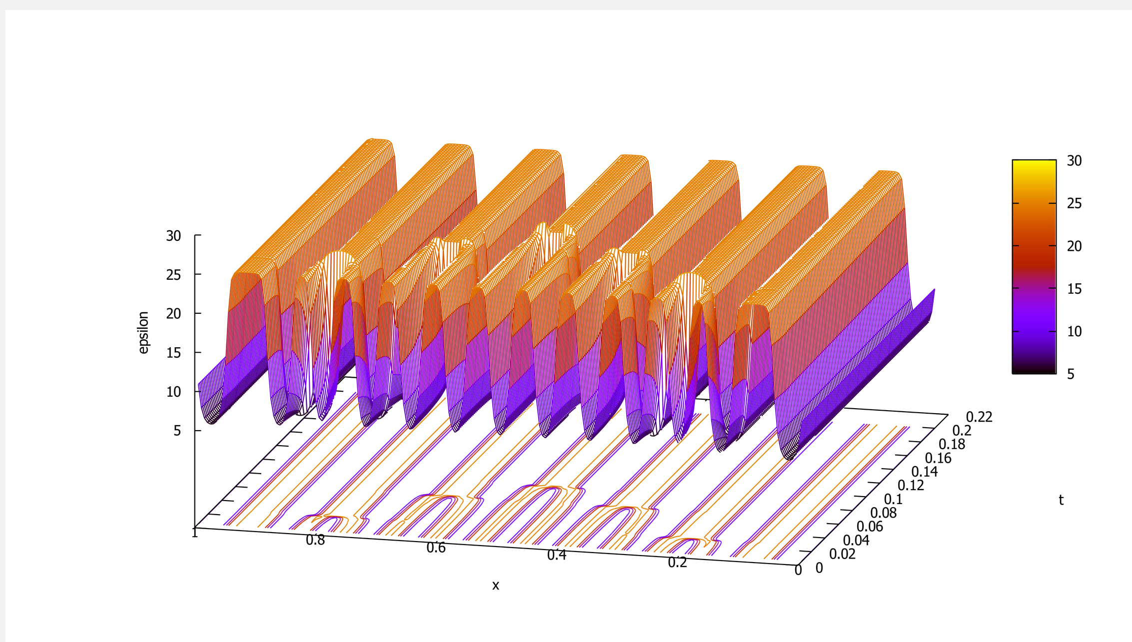


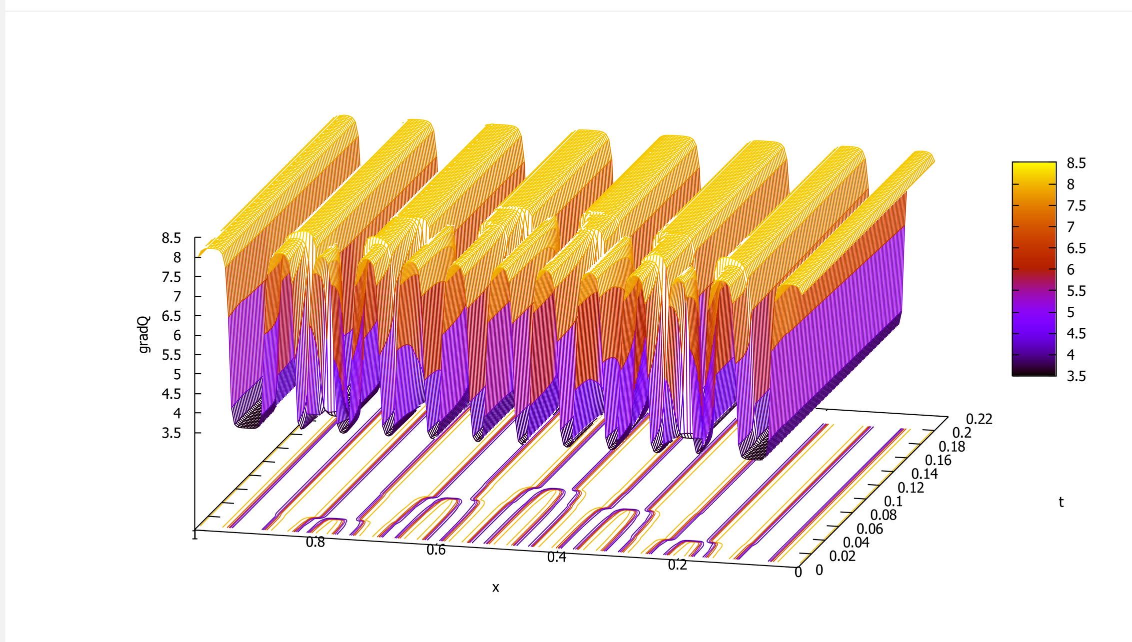
- stationary SC structure is a quasi-periodic sequence of regions with alternating upper and lower stable ε values
- time-asymptotically, this solution can be calculated analytically

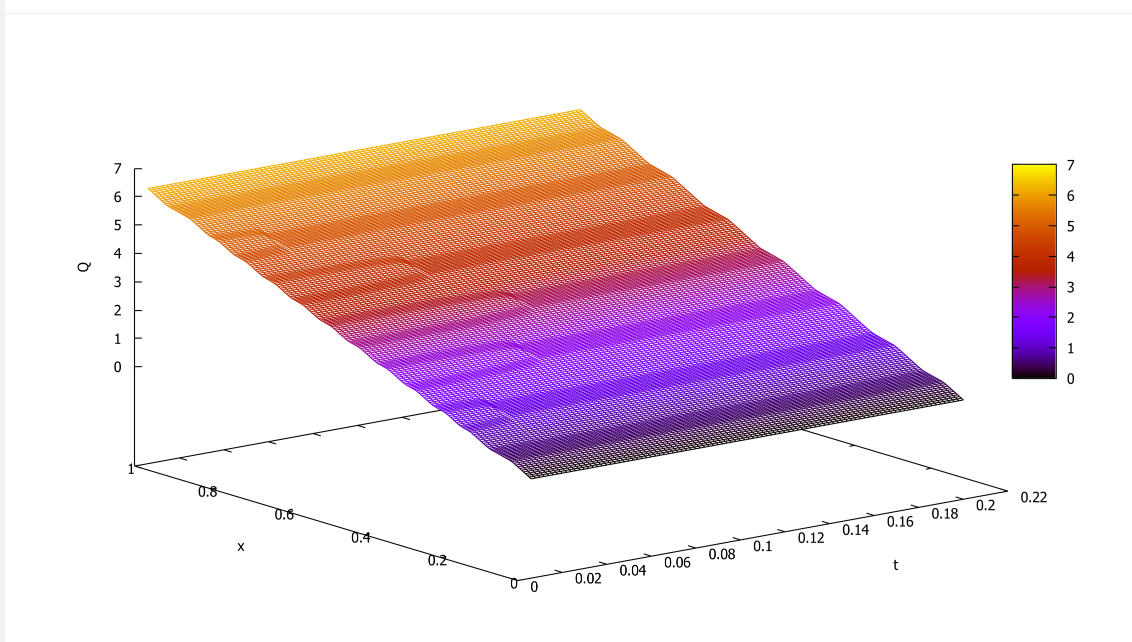


Numerical solution in long-time asymptotic regime, shown with the solid line. Exact analytic solution represented by the two branches shown with red and green squares ($\psi=Q_y/\sqrt{\varepsilon}$)

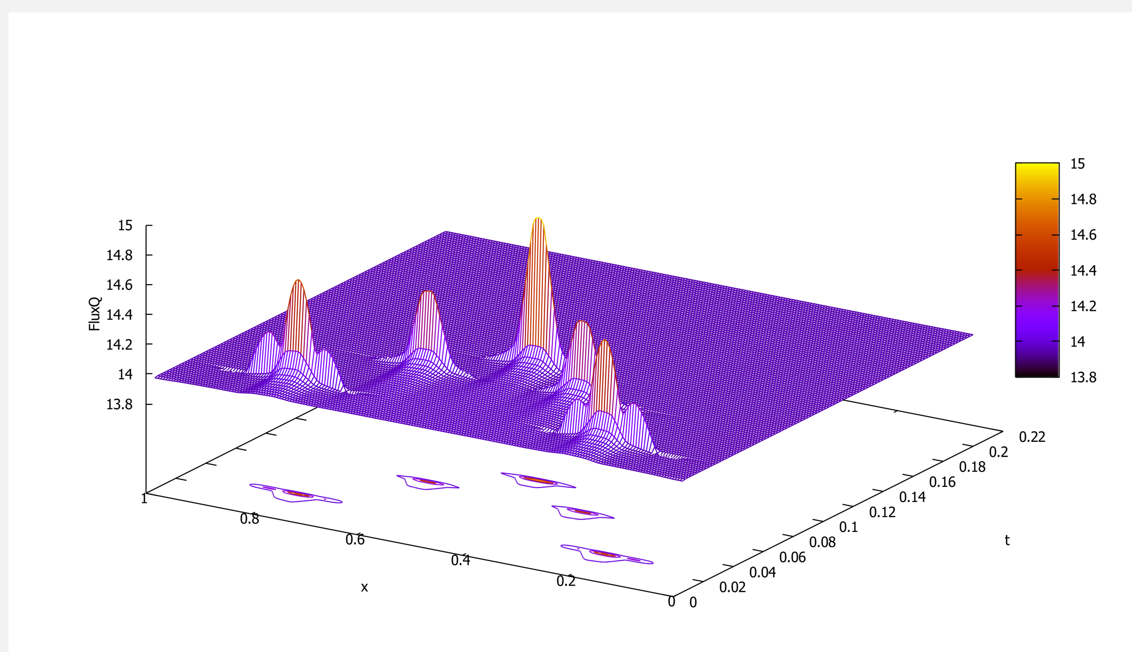
- quasi-stationary SC forms quickly ($t\ll1$) with n steps separated by shear layers with steep gradient of the mean vorticity Q_y and suppressed enstrophy level, ε
- number n is determined by the maximum growth rate
- over a longer time (but still $\ll1$), most of n steps merge with their neighbours and the total number of steps becomes $\approx n/2$. After this initial phase the staircase persists for a much longer time







- Q-flux grows rapidly, and strongly deviates from its globally constant value precisely at the merger locations
- shown is a sequence of mergers of 12 initial steps. They proceed symmetrically from the boundaries towards the centre
- process continues until the mergers converge at the centre and the central two steps merge into a bigger step



- flux remains constant when no mergers occur

$$\left[\varepsilon^{1/2}\left(1+Q_y^2/\varepsilon\right)^{-2}+D\right]Q_y\equiv b=const$$

- the flux builds up in two phases (slow and fast) before it drops abruptly to its averaged value after the merger
- the first phase is an initial growth that lasts to about $t\approx0.065$. The flux increase remains relatively small, <0.01
- the second phase is explosive and can be accurately fit by the following function,

$$F=F_0+B/\left(t_0-t\right)^{\alpha}$$

with $t_0\approx0.0863$, $B\approx0.000806$, $\alpha\approx0.879$, and a residual flux $F_0\approx-0.0171$.

References

[1] Balmforth, N. J., Smith, S. G. L. & Young, W. R. 1998 Dynamics of interfaces and layers in a stratified turbulent fluid. *Journal of Fluid Mechanics* **355**, 329–358.

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[3] Rhines, P. B. 1975 Waves and turbulence on a beta-plane. *Journal of Fluid Mechanics* **69**, 417–443.