Single Eddy Mixing in Cahn-Hilliard Flows

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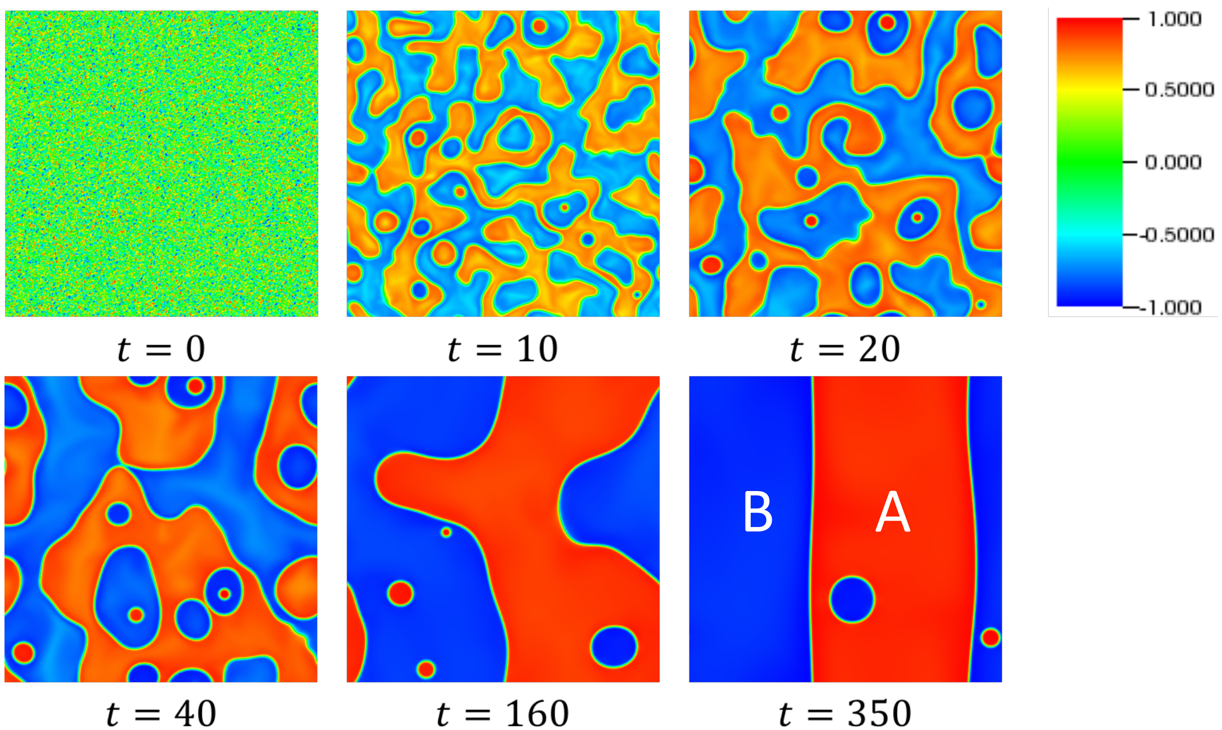
What Is the CHNS System?

- Elastic media – Fluid with internal DoFs → "springiness"
- The Cahn-Hilliard Navier-Stokes (CHNS) system describes ***phase separation*** for binary fluid (i.e. ***Spinodal Decomposition***)
- Miscible phase -> Immiscible phase



What Is the CHNS System?

- How to describe the system: the concentration field
- $\psi(\vec{r},t) \stackrel{\text{def}}{=} [\rho_A(\vec{r},t) - \rho_B(\vec{r},t)]/\rho$: scalar field
- $\psi \in [-1,1]$
- CHNS equations:

$$\partial_t \psi + \vec{v} \cdot \nabla \psi = D\nabla^2(-\psi + \psi^3 - \xi^2 \nabla^2 \psi)$$

$$\partial_t \omega + \vec{v} \cdot \nabla \omega = \frac{\xi^2}{\rho} \vec{B}_\psi \cdot \nabla \nabla^2 \psi + \nu \nabla^2 \omega$$

Why Care?

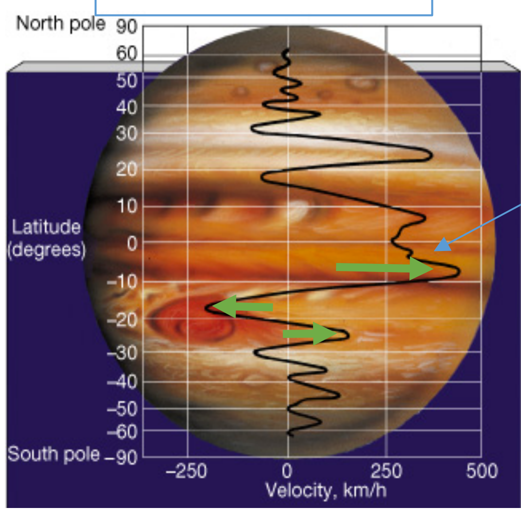
- Useful to examine familiar themes in plasma turbulence from new vantage point
- Some key issues in plasma turbulence:

1. Electromagnetics Turbulence
   - CHNS vs 2D MHD: analogous, with interesting differences.
   - Both CHNS and 2D MHD are ***elastic*** systems
   - Most systems = 2D/Reduced MHD + many linear effects
     - ➢ Physics of dual cascades and constrained relaxation → relative importance, selective decay?
     - ➢ Physics of wave-eddy interaction effects on nonlinear transfer (i.e. Alfven effect ←→ Kraichnan)

Why Care?

2. Zonal flow formation → negative viscosity phenomena → phase separation process
   - ZF can be viewed as a “spinodal decomposition” of momentum.
   - What determines scale?

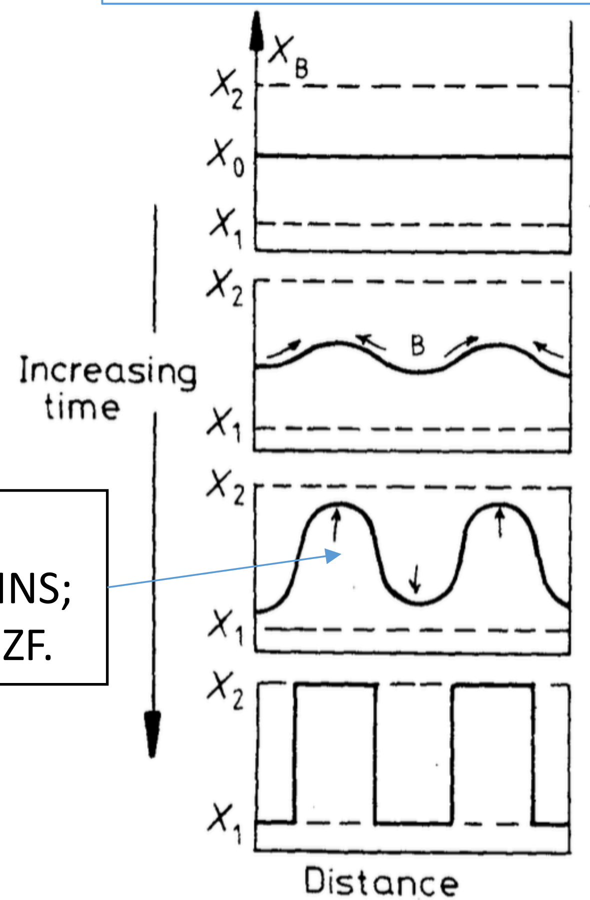
Zonal Flow



http://astronomy.nju.edu.cn/~lixd/GA/AT4/AT411/HTML/AT41102.htm

Arrows:
ψ for CHNS;
flow for ZF.

Spinodal Decomposition



[Porter 1981]

UC San Diego

Why Care?

3. “Blobby Turbulence” → how to understand blob coalescence and relation to cascades?
   - CHNS is a naturally blobby system of turbulence.
   - What is the role of structure in interaction?
   - How to understand multiple cascades of blobs and energy?

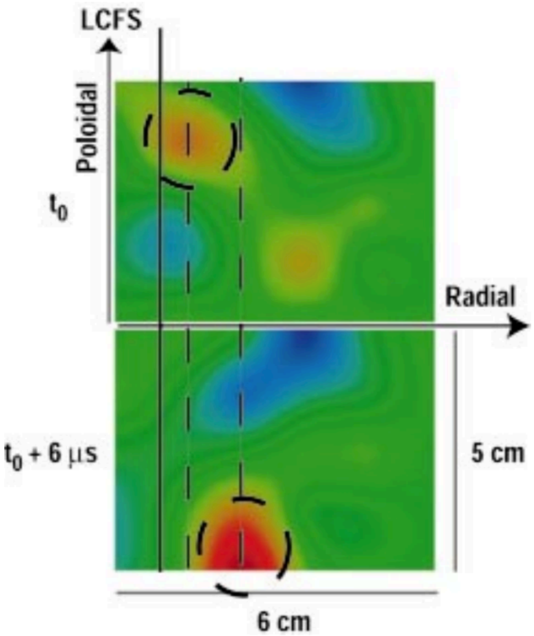


FIG. 4. (Color) Two frames from BES showing 2-D density plots. There is a time difference of 6 μs between frames. Red indicates high density and blue low density. A structure, marked with a dashed circle and shown in both frames, features poloidal and radial motion.

[J. A. Boedo et.al. 2003]

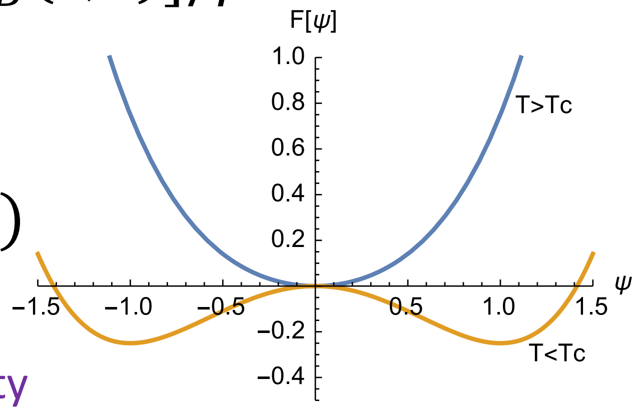
- CHNS exhibits all of the above, with many new twists

Outline

- A Brief Derivation of the CHNS Model
- 2D CHNS and 2D MHD
- Linear Wave
- Single Eddy Mixing in 2D MHD: Expulsion
- Single Eddy Mixing in 2D Cahn-Hilliard Flow
- 3 Stages
- Band Mergers
- Time Scales
- Conclusions

A Brief Derivation of the CHNS Model

- Second order phase transition → Landau Theory.
- Order parameter: $\psi(\vec{r},t) \stackrel{\text{def}}{=} [\rho_A(\vec{r},t) - \rho_B(\vec{r},t)]/\rho$
- Free energy:



$$F(\psi) = \int d\vec{r}\left(\frac{1}{2}C_1\psi^2 + \frac{1}{4}C_2\psi^4 + \frac{\xi^2}{2}|\nabla\psi|^2\right)$$

Phase Transition — Gradient Penalty

- $C_1(T), C_2(T)$.
- Isothermal $T < T_C$. Set $C_2 = -C_1 = 1$:

$$F(\psi) = \int d\vec{r}\left(-\frac{1}{2}\psi^2 + \frac{1}{4}\psi^4 + \frac{\xi^2}{2}|\nabla\psi|^2\right)$$

A Brief Derivation of the CHNS Model

- Continuity equation: $\frac{d\psi}{dt} + \nabla \cdot \vec{J} = 0.$
- Fick's Law: $\vec{J} = -D\nabla\mu.$
- Chemical potential: $\mu = \frac{\delta F(\psi)}{\delta\psi} = -\psi + \psi^3 - \xi^2\nabla^2\psi.$
- Combining → Cahn Hilliard equation:

$$\frac{d\psi}{dt} = D\nabla^2\mu = D\nabla^2(-\psi + \psi^3 - \xi^2\nabla^2\psi)$$

- $d_t = \partial_t + \vec{v} \cdot \nabla.$
- Surface tension: force in Navier-Stokes equation:

$$\partial_t\vec{v} + \vec{v} \cdot \nabla\vec{v} = -\frac{\nabla p}{\rho} - \psi\nabla\mu + \nu\nabla^2\vec{v}$$

- For incompressible fluid, $\nabla \cdot \vec{v} = 0.$

2D CHNS and 2D MHD

	2D MHD	2D CHNS
Magnetic Potential	A	ψ
Magnetic Field	$\mathbf{B}$	$\mathbf{B}_\psi$
Current	j	j_ψ
Diffusivity	η	D
Interaction strength	$\frac{1}{\mu_0}$	ξ^2

- 2D CHNS Equations:

$$\partial_t \psi + \vec{v} \cdot \nabla \psi = D\nabla^2(-\psi + \psi^3 - \xi^2 \nabla^2 \psi)$$

$$\partial_t \omega + \vec{v} \cdot \nabla \omega = \frac{\xi^2}{\rho} \vec{B}_\psi \cdot \nabla \nabla^2 \psi + \nu \nabla^2 \omega$$

$-\psi$: Negative diffusion term
ψ^3: Self nonlinear term
$-\xi^2\nabla^2\psi$: Hyper-diffusion term

With $\vec{v} = \hat{z} \times \nabla \phi$, $\omega = \nabla^2 \phi$, $\vec{B}_\psi = \hat{z} \times \nabla \psi$, $j_\psi = \xi^2 \nabla^2 \psi$. $\psi \in [-1,1]$.

- 2D MHD Equations:

$$\partial_t A + \vec{v} \cdot \nabla A = \eta \nabla^2 A$$

$$\partial_t \omega + \vec{v} \cdot \nabla \omega = \frac{1}{\mu_0 \rho} \vec{B} \cdot \nabla \nabla^2 A + \nu \nabla^2 \omega$$

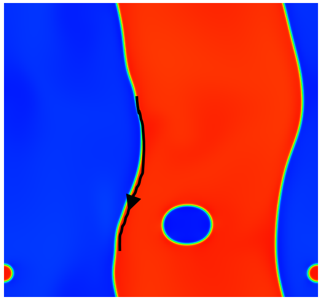
A: Simple diffusion term

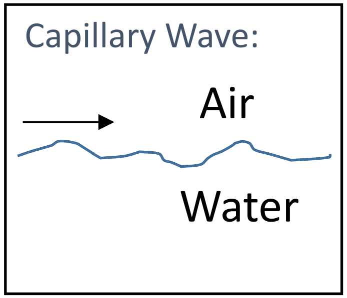
With $\vec{v} = \hat{z} \times \nabla \phi$, $\omega = \nabla^2 \phi$, $\vec{B} = \hat{z} \times \nabla A$, $j = \frac{1}{\mu_0} \nabla^2 A$

Linear Wave

- CHNS supports linear "elastic" wave:

$$\omega(k) = \pm\sqrt{\frac{\xi^2}{\rho}}|\vec{k}\times\vec{B}_{\psi 0}| - \frac{1}{2}i(CD+\nu)k^2$$

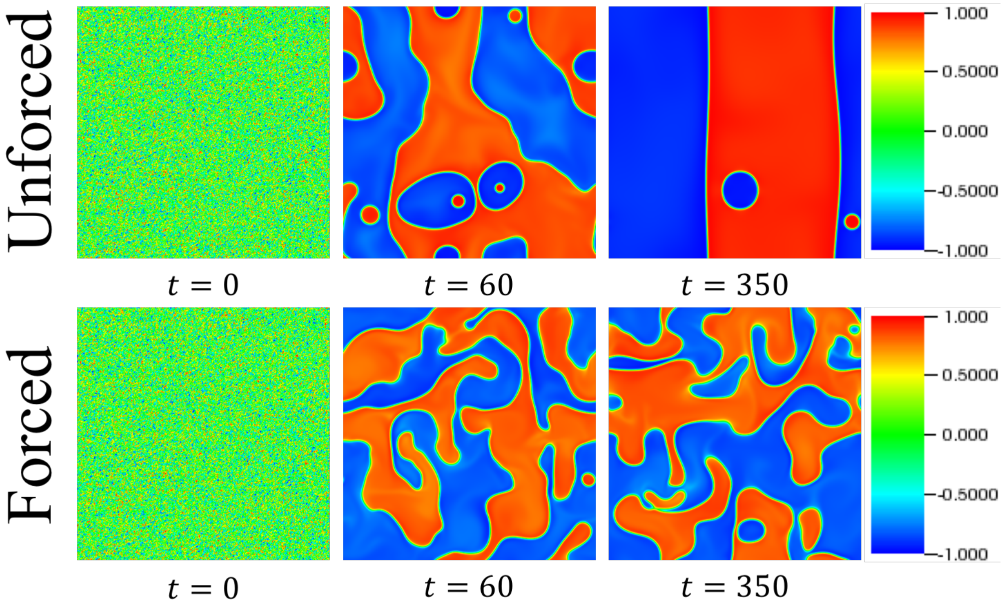


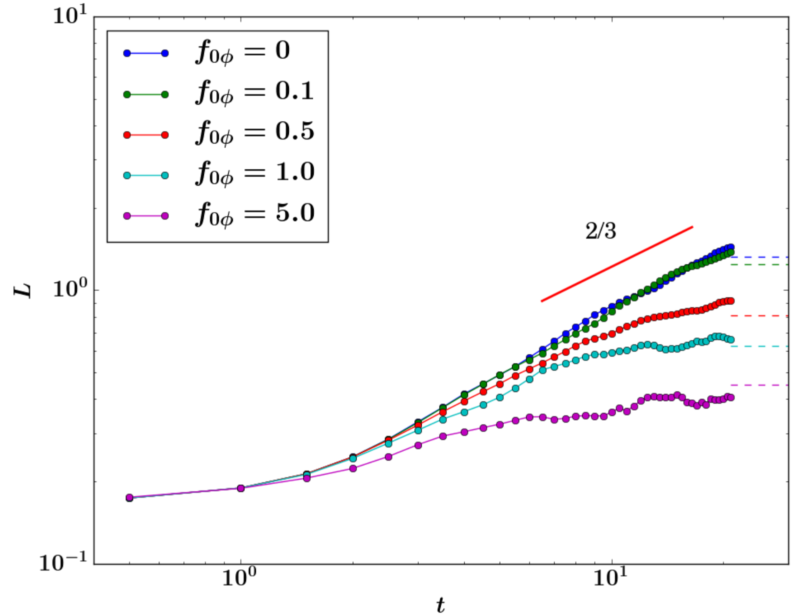


Where $C \equiv [-1 - 6\psi_0\nabla^2\psi_0/k^2 - 6(\nabla\psi_0)^2/k^2 - 6\psi_0\nabla\psi_0\cdot i\mathbf{k}/k^2 + 3\psi_0^2 + \xi^2k^2]$

- Akin to capillary wave at phase interface.
- Propagates ***only*** along the interface of the two fluids, where $|\vec{B}_\psi| = |\nabla\psi| \neq 0$.
- Analogue of Alfven wave.
- Important differences:
  - $\vec{B}_\psi$ in CHNS is large only in the interfacial regions.
  - Elastic wave activity does not fill space.

Results on CHNS Turbulence Study





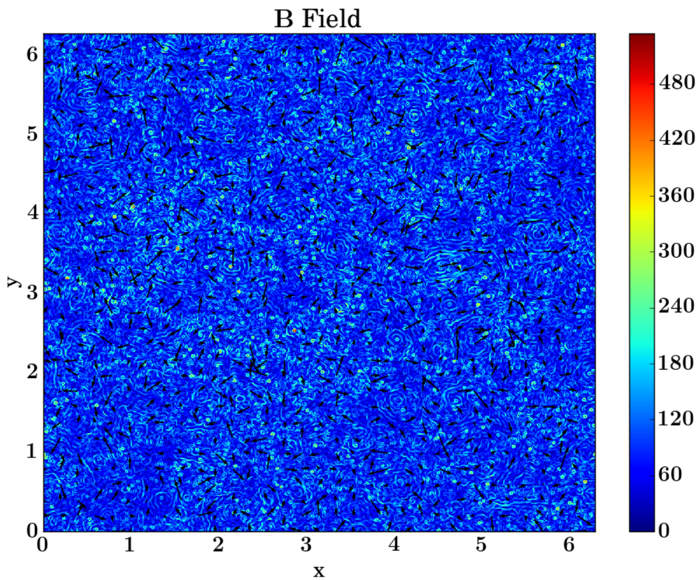
- Fluid forcing → Fluid straining vs Blob coalescence
- Scale where turbulent straining ~ elastic restoring force (due surface tension): Hinze Scale

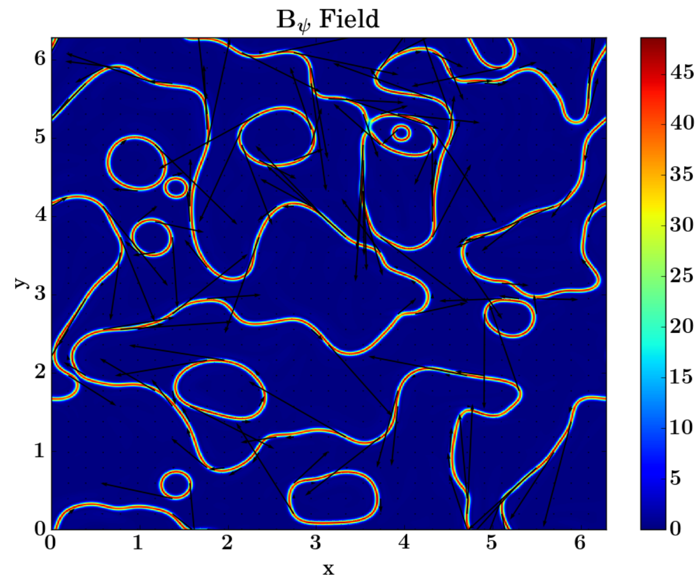
$$L_H \sim \left(\frac{\rho}{\xi}\right)^{-1/3} \epsilon_\Omega^{-2/9}$$

Results on CHNS Turbulence Study

- Dual cascade:
  - ***Inverse*** cascade of $\langle \psi^2 \rangle$
  - ***Forward*** cascade of E
- $k^{-7/3}$ for $\langle \psi^2 \rangle$ spectrum
- k^{-3} for kinetic energy spectrum (not -3/2!)
- Interface packing matters.

MHD

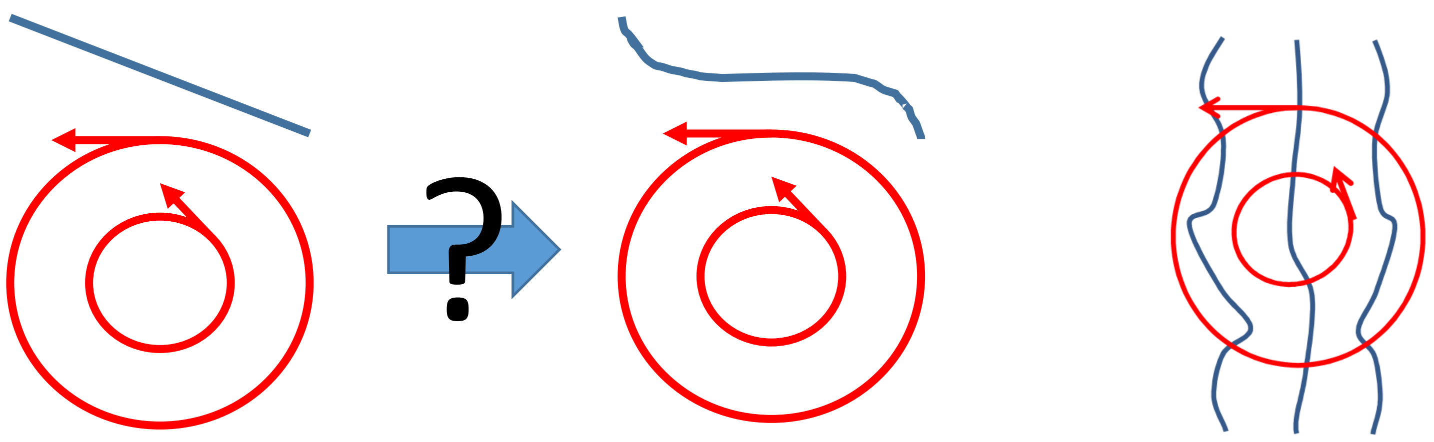




CHNS

Why Single Eddy Mixing?

- Structures are the key → need understand how a ***single eddy*** interacts with ψ field
- Mixing of $\nabla\psi$ by a single eddy → characteristic time scales?
- Evolution of structure?
- Analogous to flux expulsion in MHD (Weiss, '66)

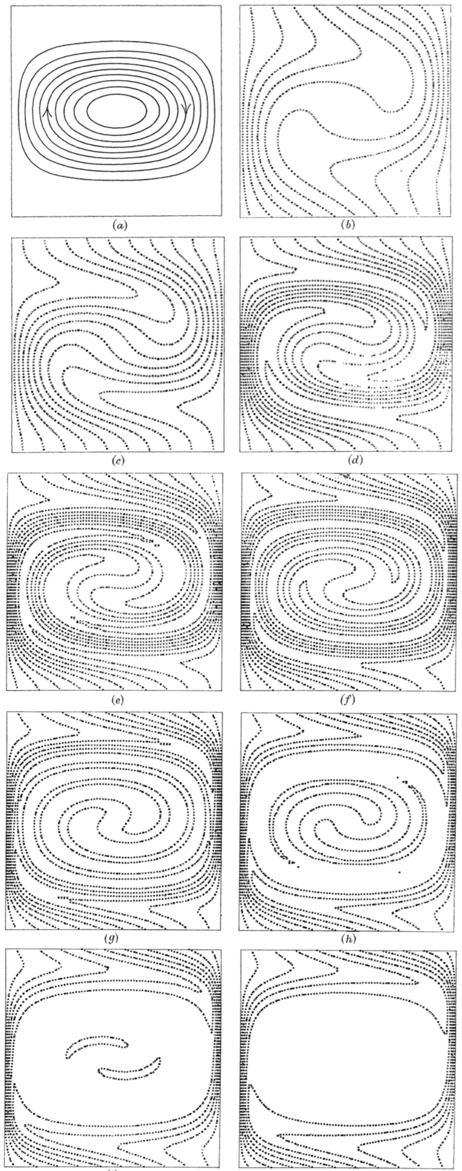


Single Eddy Mixing in 2D MHD: Expulsion

- When a convection eddy is imposed in a weak magnetic field, the magnetic field is expelled and amplified outside the eddy.
- This is called flux expulsion.
- The equation (kinematic, i.e. back reaction is ignored):

$$\partial_t A + \vec{v} \cdot \nabla A = \eta \nabla^2 A$$

- Also relevant to PV homogenization → ZF



Weiss 1966

Single Eddy Mixing in 2D MHD: Expulsion

- Main results of Weiss 1966 on Expulsion:
  - The final value of <B²> can be estimated by $\langle B^2\rangle \sim Rm^{1/2} B_0^2$
  - The time for <B²> to reach a steady state is $\tau \sim Rm^{1/3}\tau_0$

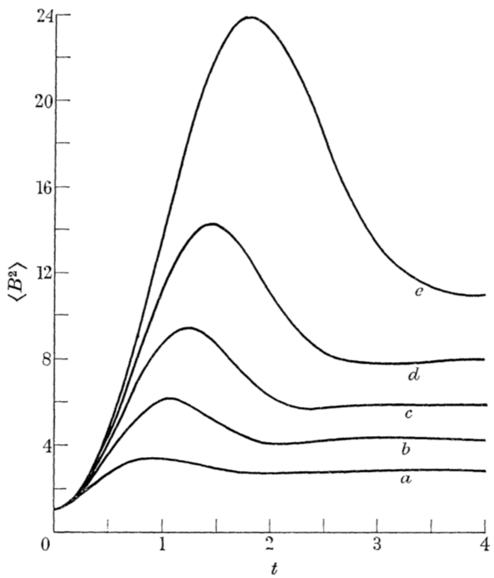


FIGURE 5. Magnetic energy as a function of time. Curves labelled a, b, c, d, e have R_m = 40, 100, 200, 400, 1000 respectively.

Weiss 1966

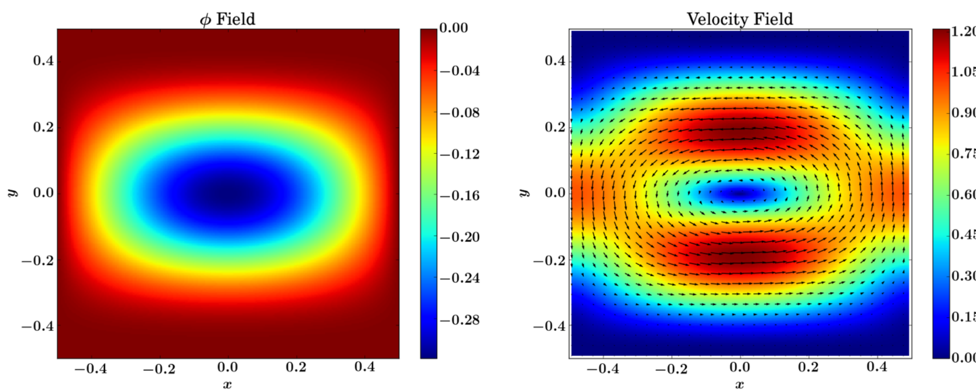
- Main results of Rhines and Young 1983 on PV Homogenization:
  - Two stages: rapid and slow
  - Rapid stage: dominated by shear-augmented diffusion, with time scale $\tau_{mix} \sim Rm^{1/3}\tau_0$
  - Slow stage: usual diffusion, with time scale $\tau_{slow} \sim Rm\ \tau_0$

Single Eddy Mixing in 2D Cahn-Hilliard Flow

- The equation (kinematic, i.e. back reaction is ignored):

$$\partial_t \psi + \vec{v} \cdot \nabla \psi = D\nabla^2(-\psi + \psi^3 - \xi^2 \nabla^2 \psi)$$

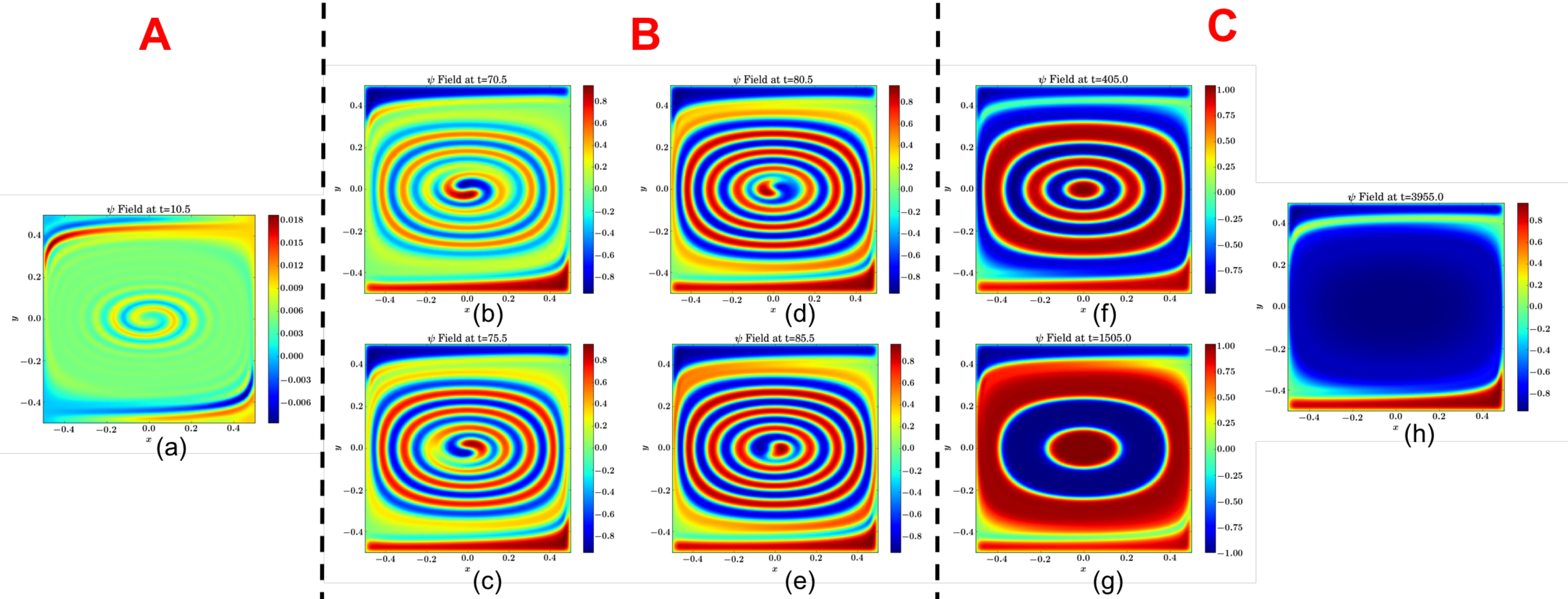
- Initial condition (analogous to Weiss 1966): $\vec{B}_0$ + eddy:



- Boundary condition: Dirichlet boundaries in both directions: $\psi = \psi_0$ and $\nabla^2\psi = \nabla^2\psi_0$ at boundaries.
- Dimensionless numbers:
  - $Pe = L_0 v_0 / D$ (Pe←→Rm)
  - $Ch = \xi L_0$

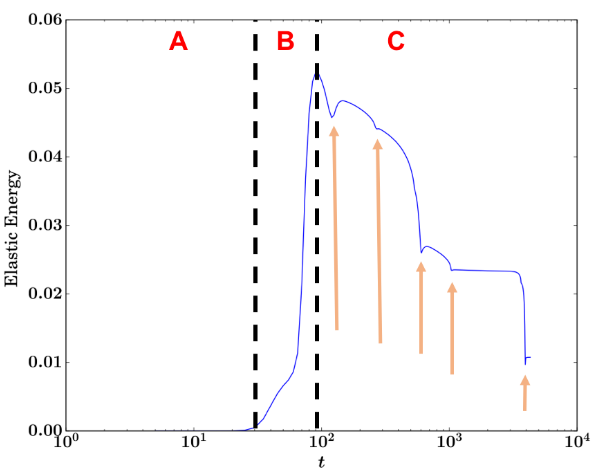
3 Stages

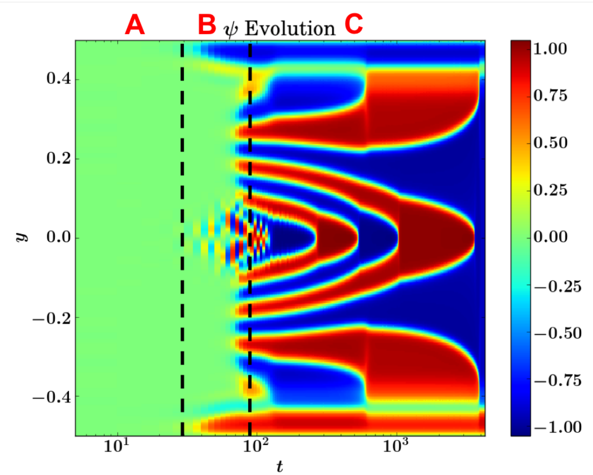
- 3 stages: (A) the *"jelly roll"* stage, (B) the *topological evolution* stage, and (C) the *target pattern* stage.
- ψ ultimately homogenized in slow time scale, but metastable target patterns formed and merge.

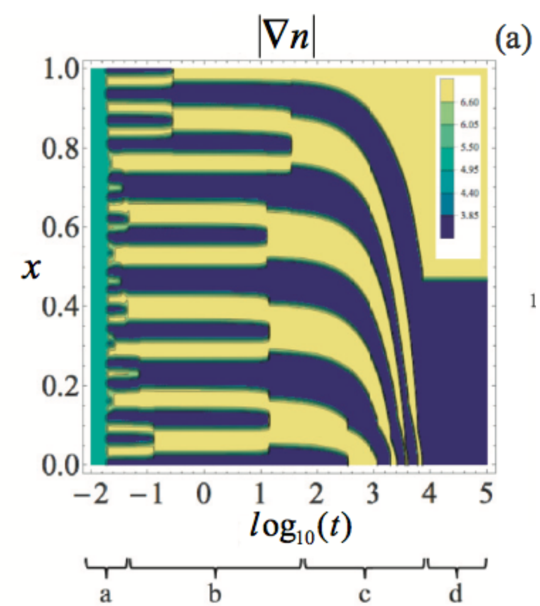


Band Merger

- The bands merge on a time scale ***<u>exponentially long</u>*** relative to eddy turnover time.
- The 3 stages are reflected in the elastic energy plot.
- The target bands mergers are related to the dips in the target pattern stage.
- The band merger process is similar to the step merger in drift-ZF staircases.



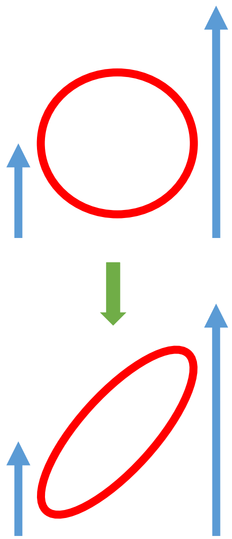




Ashourvan *et.al.* 2016

Time Scales

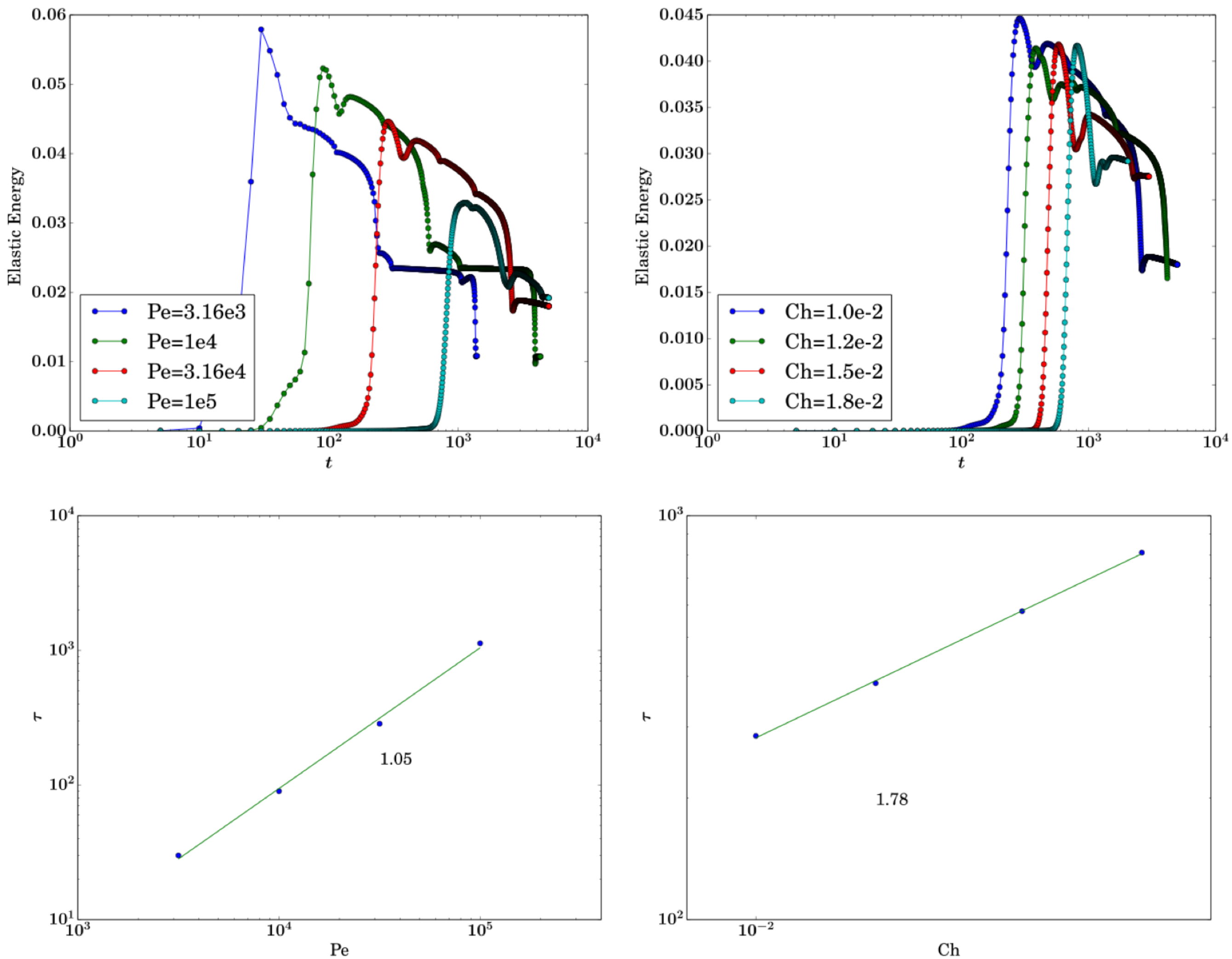
- Analogous to the $Rm^{1/3}$ time scale in MHD, the mixing time scale of the shear + dissipation hybrid case is $\tau_{mix} \sim Pe^{1/5} Ch^{-2/5} t_0$.
- Brief derivation:
  - CH equation → $\langle \delta r^4 \rangle \sim D\xi^2 t$
  - Relate δr and δy according to shear s:
  $$\frac{d}{dt}\delta y \sim s\delta r$$
  - So $\langle \delta y^4 \rangle \sim s^4 D\xi^2 t^5$.
  - Note that $Pe \sim L_y v/D$
  - So $\tau_{mix} \sim Pe^{1/5} Ch^{-2/5} t_0$



Time Scales

- Time to reach the maximum elastic energy:

$$\tau_m \sim PeCh^2 t_0$$



Conclusions

- Even kinematic single eddy mixing can exhibit unexpected nontrivial dynamics.
- 3 stages: (A) the *"jelly roll"* stage, (B) the *topological evolution* stage, and (C) the *target pattern* stage.
- Band merger process occurs on a time scale exponentially long relative to the eddy turnover time.
- Band merger process resembles step merger in drift-ZF staircases.
- Multi time-scale process: the $Pe^{1/5}$ and the Pe^{1} time scale.