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Cascades, 'Blobby' Turbulence, and Target Pattern Formation in Elastic Systems

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Outline

- What Is Spinodal Decomposition?
- Why should a plasma physicist care? -- Connections
- The Cahn-Hilliard Navier-Stokes (CHNS) Model
- Linear elastic wave & Ideal quadratic conserved quantities
- Dual Cascades
- Important length scales and ranges
- Power laws
- Single Eddy: Flux Expulsion -> Target Pattern
- Future Plans

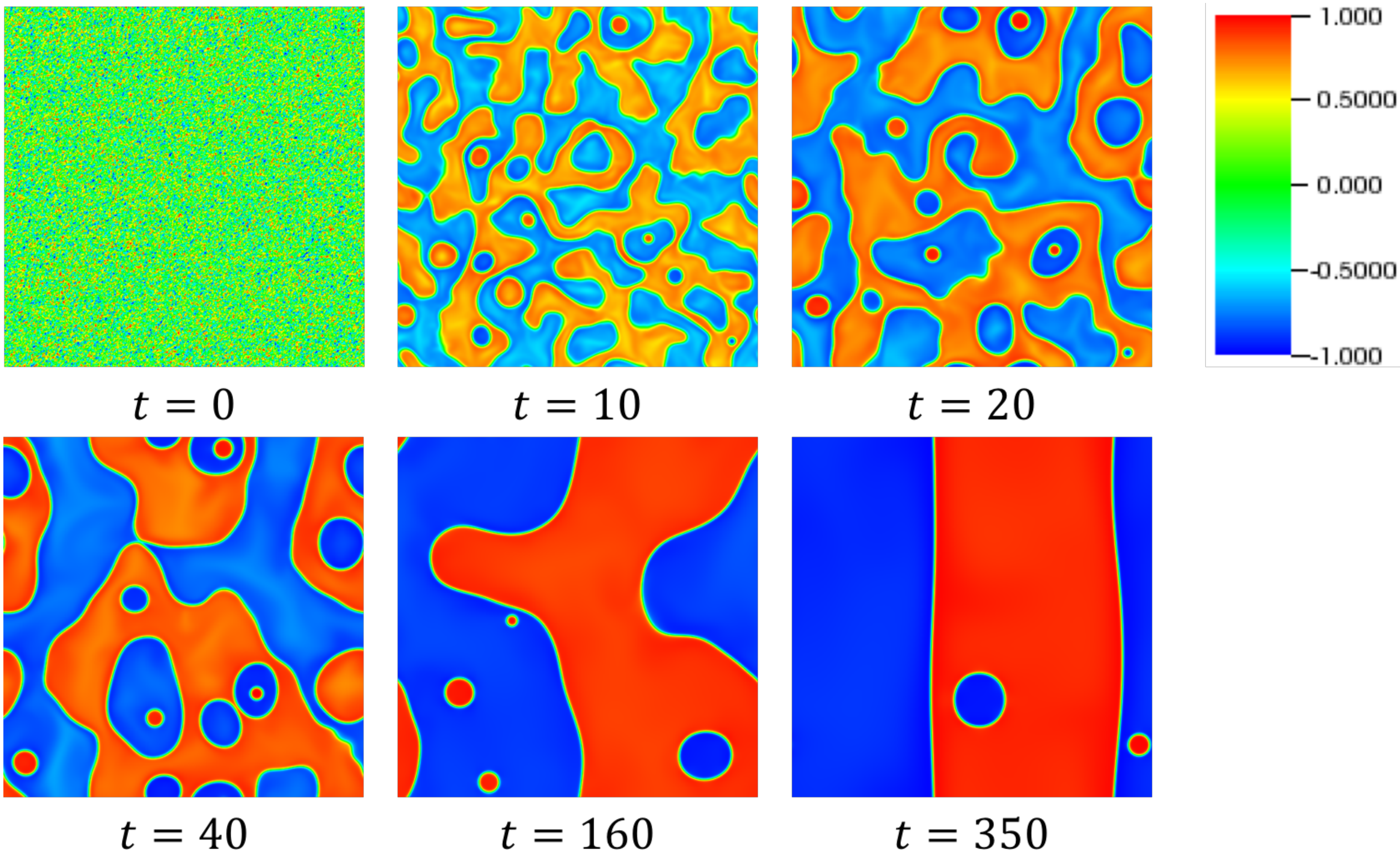
What Is Spinodal Decomposition?

- This is not a traditional study of plasma physics. It is about the fundamental physics of cascades and self-organization in ***elastic turbulence systems***. Examples of elastic system: ***MHD & spinodal decomposition.*** Spinodal decomposition is related to ***blobby turbulence*** and ***zonal flow***.
- Spinodal decomposition is a 2^{nd} order phase transition for binary fluid mixture.
- Miscible phase -> Immiscible phase
- For example, at high enough temperature, water and oil can form a single thermodynamic phase, and when it's cooled down, the separation of oil-rich and water-rich phases occurs.
- The order parameter: the local relative concentration field:

$$\psi(\vec{r},t) \stackrel{\text{def}}{=} [\rho_A(\vec{r},t) - \rho_B(\vec{r},t)]/\rho$$

- $\psi = 1$ means A-rich, $\psi = -1$ means B-rich.

What Is Spinodal Decomposition?



Why Care?

- In the edge of Tokamaks, localized density fluctuation structure or "blobs" flow together and transport large amounts of edge plasma. This phenomenon is called "blobby" turbulence.
- Blobby turbulence is important to understand edge physics in Tokamaks, yet still is not understood. What is the criterion for "blobbyness"?

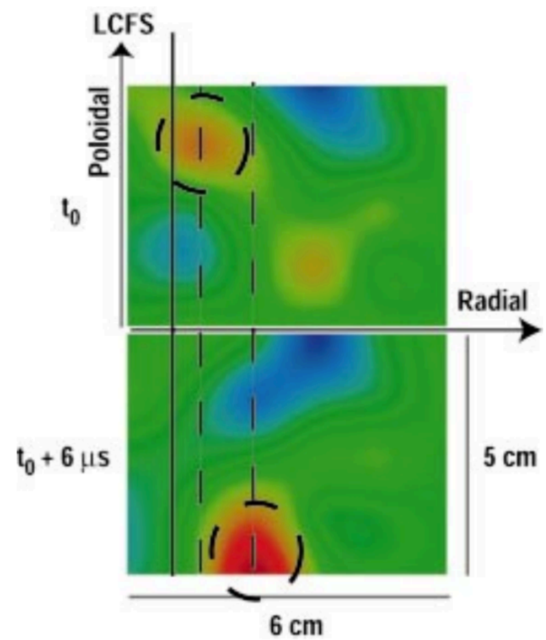


FIG. 4. (Color) Two frames from BES showing 2-D density plots. There is a time difference of 6 μs between frames. Red indicates high density and blue low density. A structure, marked with a dashed circle and shown in both frames, features poloidal and radial motion.

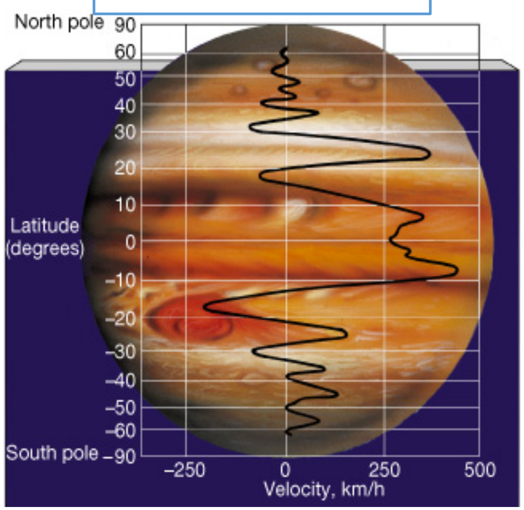
[J. A. Boedo et.al. 2003]

- ***<u>Spinodal decomposition is naturally blobby turbulence system.</u>*** It can provide insight for fusion studies.
  - ➢Role of structure in interaction
  - ➢Multiple cascades of blobs and energy.

Why Care?

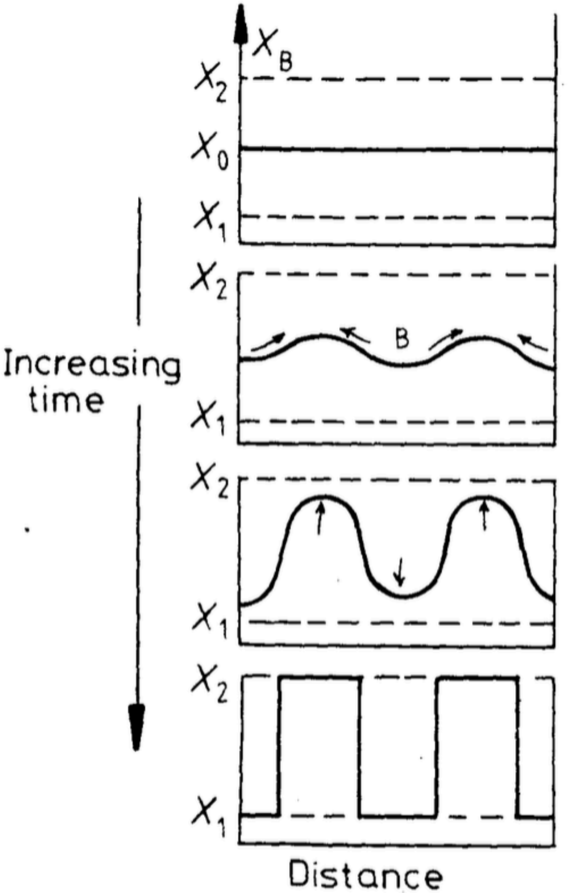
- Spinodal decomposition is also related to zonal flow. ***ZF can be viewed as a "spinodal decomposition" of momentum.***
- Pattern formation follows from negative diffusion/negative viscosity.

Zonal Flow



http://astronomy.nju.edu.cn/~lixd/GA/AT4/AT411/HTML/AT41102.htm

Spinodal Decomposition

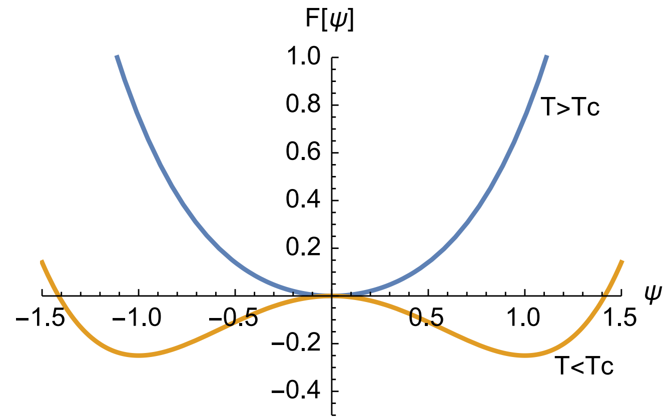


[Porter 1981]

The Cahn-Hilliard Navier-Stokes (CHNS) Model

- Landau Theory: the free energy functional:

$$F(\psi) = \int d\vec{r}\Big(\underbrace{-\frac{1}{2}\psi^2 + \frac{1}{4}\psi^4}_{\text{Phase Transition}} + \underbrace{\frac{\xi^2}{2}|\nabla\psi|^2}_{\text{Gradient Penalty}}\Big)$$



- Chemical potential: $\mu = \frac{\delta F(\psi)}{\delta\psi} = -\psi + \psi^3 - \xi^2\nabla^2\psi$. Fick's Law: $\vec{J} = -D\nabla\mu$. Continuity equation: $\frac{d\psi}{dt} + \nabla\cdot\vec{J} = 0$. Combining the above together:

$$\frac{d\psi}{dt} = D\nabla^2\mu = D\nabla^2(-\psi + \psi^3 - \xi^2\nabla^2\psi)$$

- The fluid velocity comes in via the convection term $d_t = \partial_t + \vec{v}\cdot\nabla$.
- The surface tension enters the Navier-Stokes Equation as a force:

$$\partial_t\vec{v} + \vec{v}\cdot\nabla\vec{v} = -\frac{\nabla p}{\rho} - \psi\nabla\mu + \nu\nabla^2\vec{v}$$

The Cahn-Hilliard Navier-Stokes (CHNS) Model

- For incompressible fluid, it is convenient to take the curl and work with vorticity.
- The 2D CHNS Equations:

$$\boxed{\begin{aligned} \partial_t \psi + \vec{v} \cdot \nabla \psi &= D \nabla^2 (-\psi + \psi^3 - \xi^2 \nabla^2 \psi) \\ \partial_t \omega + \vec{v} \cdot \nabla \omega &= \frac{\xi^2}{\rho} \vec{B}_\psi \cdot \nabla \nabla^2 \psi + \nu \nabla^2 \omega \end{aligned}}$$

$-\psi$: Negative diffusion term
ψ^3: Self nonlinear term
$-\xi^2 \nabla^2 \psi$: Hyper-diffusion term

With $\vec{v} = \hat{z} \times \nabla \phi$, $\omega = \nabla^2 \phi$, $\vec{B}_\psi = \hat{z} \times \nabla \psi$, $j_\psi = \xi^2 \nabla^2 \psi$

- The 2D MHD Equations:

$$\boxed{\begin{aligned} \partial_t A + \vec{v} \cdot \nabla A &= \eta \nabla^2 A \\ \partial_t \omega + \vec{v} \cdot \nabla \omega &= \frac{1}{\mu_0 \rho} \vec{B} \cdot \nabla \nabla^2 A + \nu \nabla^2 \omega \end{aligned}}$$

A: Simple diffusion term

With $\vec{v} = \hat{z} \times \nabla \phi$, $\omega = \nabla^2 \phi$, $\vec{B} = \hat{z} \times \nabla A$, $j = \frac{1}{\mu_0} \nabla^2 A$

- The force on fluid: $\frac{\xi^2}{\rho} \vec{B}_\psi \cdot \nabla \nabla^2 \psi \Longleftrightarrow \frac{1}{\mu_0 \rho} \vec{B} \cdot \nabla \nabla^2 A$
- Note that the magnetic potential A is a scalar in 2D.

	2D MHD	2D CHNS
Magnetic Potential	A	ψ
Magnetic Field	$\mathbf{B}$	$\mathbf{B}_\psi$
Current	j	j_ψ
Diffusivity	η	D
Interaction strength	$\frac{1}{\mu_0}$	ξ^2

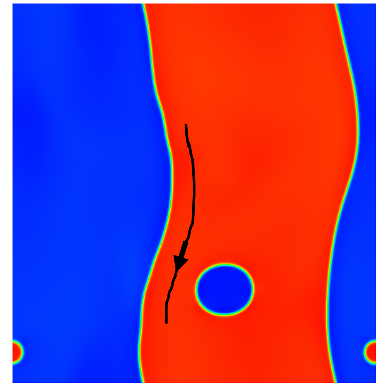
Linear Elastic Wave

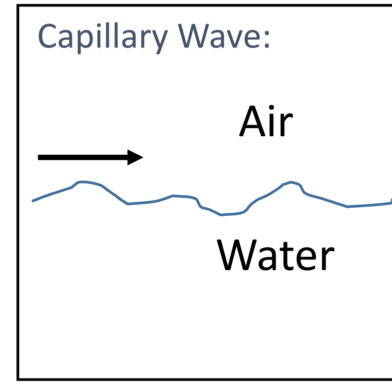
- Alfven wave in 2D MHD:

$$\omega(k) = \pm\sqrt{\frac{1}{\mu_0\rho}}\left|\vec{k}\times\vec{B}_0\right| - \frac{1}{2}i(\eta+\nu)k^2$$

- Linear elastic wave in 2D CHNS:

$$\omega(k) = \pm\sqrt{\frac{\xi^2}{\rho}}\left|\vec{k}\times\vec{B}_{\psi 0}\right| - \frac{1}{2}i(CD+\nu)k^2$$





Where $C \equiv [-1 - 6\psi_0\nabla^2\psi_0/k^2 - 6(\nabla\psi_0)^2/k^2 - 6\psi_0\nabla\psi_0\cdot i\mathbf{k}/k^2 + 3\psi_0^2 + \xi^2k^2]$

- The linear elastic wave in 2D CHNS is like a capillary wave: it only propagates along the boundary of the two fluids, where the gradient of concentration $B_\psi \neq 0$. Surface tension generates restoring force.
- The wave is similar to Alfven wave: they have similar dispersion relation; they both propagates along B field lines; both magnetic field and surface tension act like an elastic restoring force.
- Important difference: $\vec{B}$ fills the whole space; $\vec{B}_\psi$ is large only in the interface regions.

Ideal quadratic conserved quantities

- **2D MHD**

1. Energy

$$E = E^K + E^B = \int (\frac{v^2}{2} + \frac{B^2}{2\mu_0}) d^2x$$

2. Mean Square Magnetic Potential

$$H^A = \int A^2 \, d^2x$$

3. Cross Helicity

$$H^C = \int \vec{v} \cdot \vec{B} d^2x$$

- **2D CHNS**

1. Energy

$$E = E^K + E^B = \int (\frac{v^2}{2} + \frac{\xi^2 B_\psi^2}{2}) d^2x$$

2. Mean Square Concentration

$$H^\psi = \int \psi^2 \, d^2x$$

3. Cross Helicity

$$H^C = \int \vec{v} \cdot \vec{B}_\psi \, d^2x$$

- "Ideal" here means $D, \eta = 0; \nu = 0.$

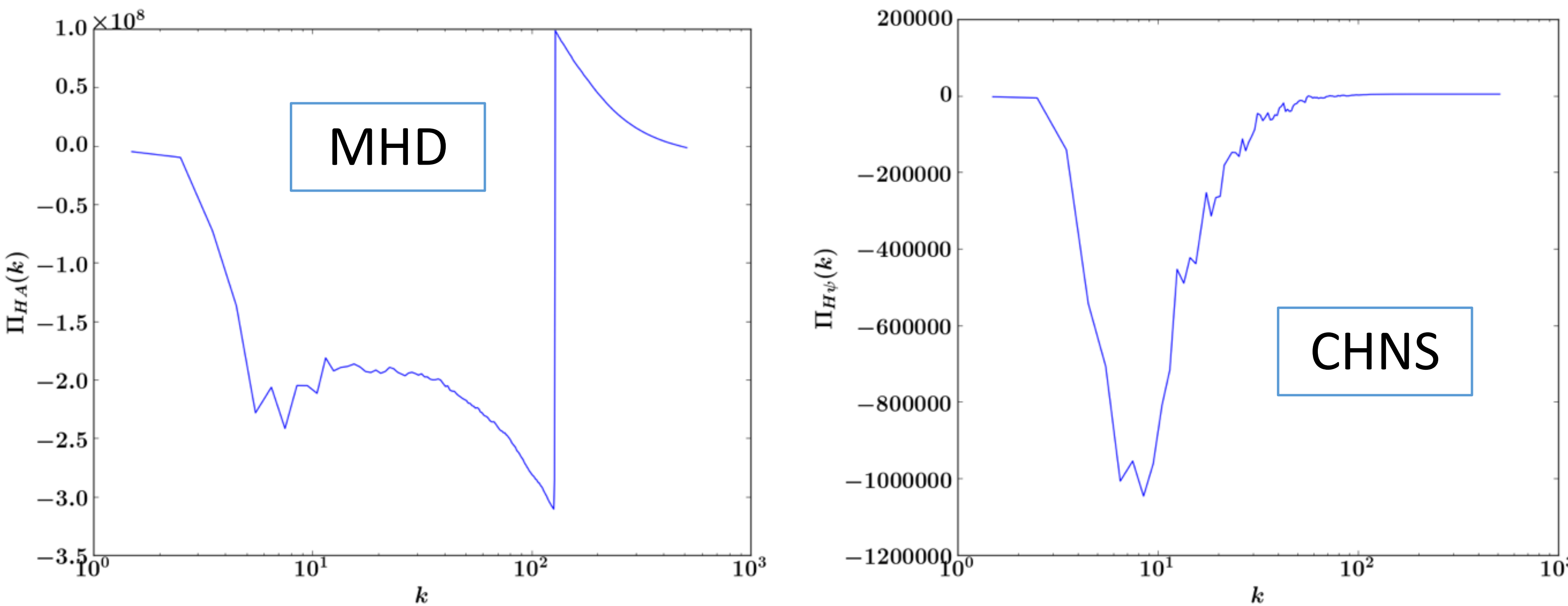
Dual Cascades

- Turbulence cascade directions are suggested by the absolute equilibrium distributions.
- The peak of the absolute equilibrium distribution for each quadratic conserved quantity is a good indicator of the corresponding cascade direction.
- The spectrum is peaked at high k -> excitation relaxes towards high k -> direct cascade.
- The spectrum is peaked at small k -> excitation relaxes towards small k -> inverse cascade.
- This approach only depends on the ideal quadratic conserved quantities of the system, so we can then obtain an indication of the cascade directions in 2D CHNS by changing the name in variables.

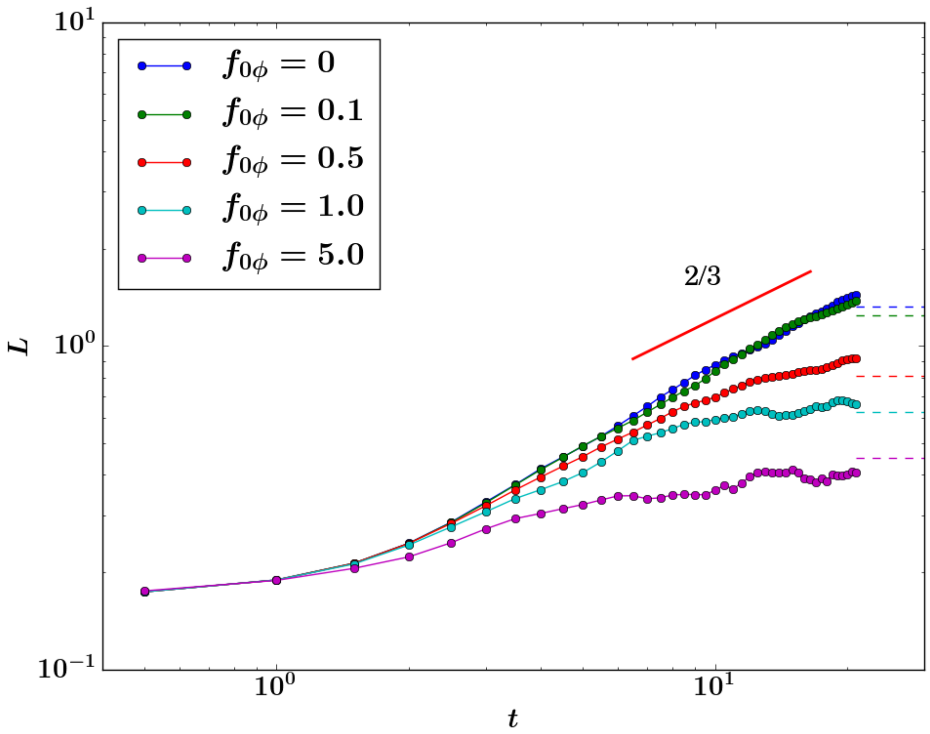
Physics System	Conserved Quantity	Cascade Direction
2D MHD	E_k	Direct
	H_k^A	Inverse
2D CHNS	E_k	Direct
	H_k^ψ	Inverse
2D NS	E_k^K	Inverse
	Ω_k	Direct

Dual Cascades

- The sign of fluxes can indicate cascades directions as well.
- The spectral fluxes are negative, and this indicates inverse cascade of H^A and H^ψ.
- For the MHD case (left), an external forcing on the magnetic potential A is applied on k = 128. The small scale A forcing drives an inverse transfer of H^A. For



Important length scales and ranges



- We verified the blob size growth is $L(t) \sim t^{2/3}$; and the growth can be arrested. The saturation length scale is consistent with the Hinze scale (dashed line).

Important length scales and ranges

- The statistically stable blob size is modelled by the Hinze scale.
- **Hinze scale**: the balance between blob merger and blob breakup processes, i.e. between turbulent kinetic energy and surface tension energy.
- In the 2D NS direct enstrophy cascade regime, the velocity distribution is $\frac{\langle v^2 \rangle}{k_H} \sim \epsilon_\Omega^{2/3} k_H^{-3}$ (It will be explained later why -3). So

$$L_H \sim (\frac{\rho}{\xi})^{-1/3} \epsilon_\Omega^{-2/9}$$

- We define the scales between L_H and dissipation scale L_d to be **the elastic range**, where the blob coalescence process dominates.
- Define a dimensionless number: $\frac{L_H}{L_d} = Hd \sim (\frac{\rho}{\xi})^{-1/3} \nu^{-1/2} \epsilon_\Omega^{-1/18}$
- $\frac{L_H}{L_d} = Hd \gg 1$ is required to form a long enough elastic range.

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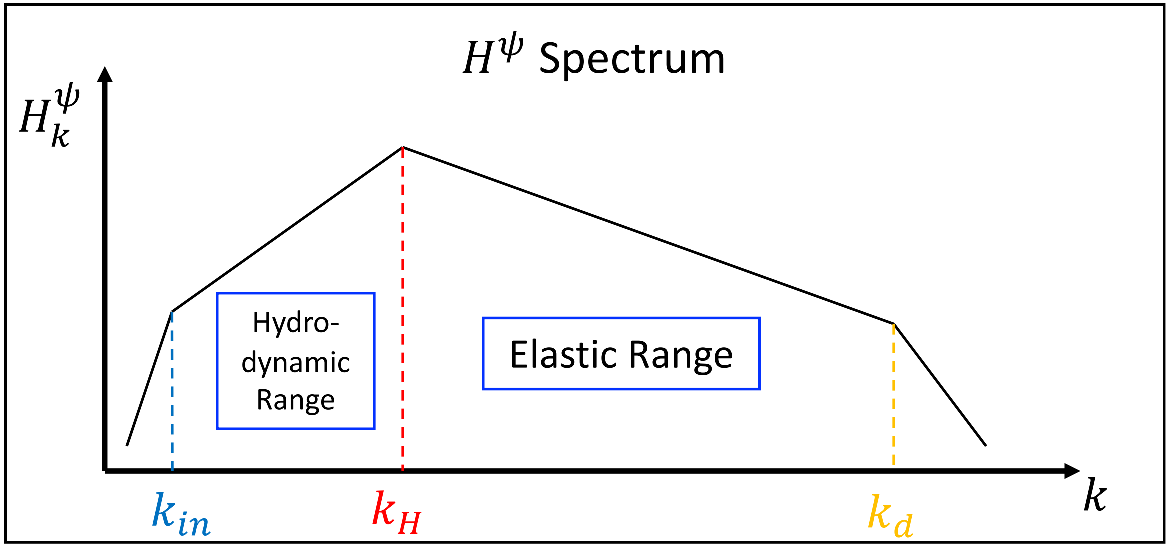
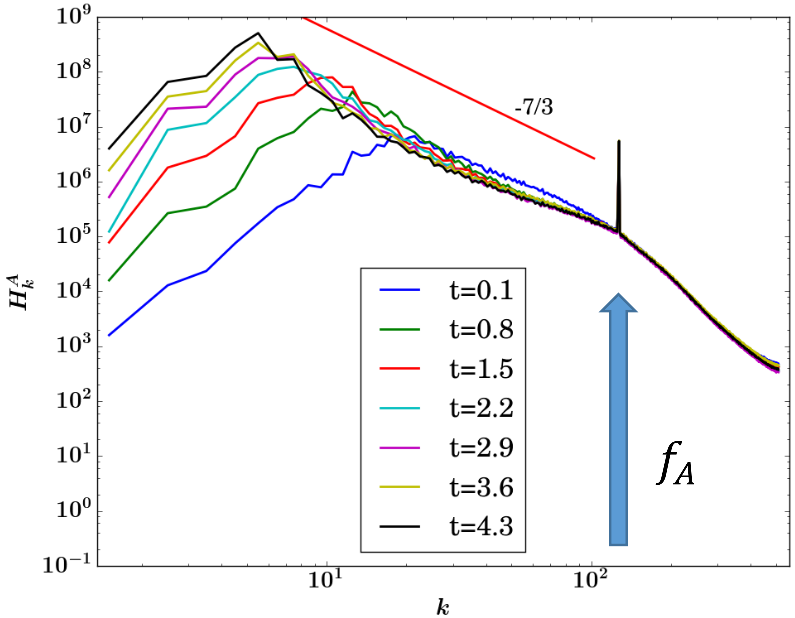
Important length scales and ranges



- In the elastic range of the 2D CHNS system, the blob coalescence process is analogous to the magnetic flux coalescence process in 2D MHD.
- The former leads to the inverse cascade of H^{ψ}, and the latter leads to the inverse cascade of H^{A} .
- In the elastic range of the 2D CHNS system, surface tension induces elasticity and plays a major role in defining a restoring force. Similarly, in 2D MHD, the magnetic field induces elasticity and make MHD different from a pure NS fluid.
- The 2D CHNS system is more MHD-like in the elastic range.

H_k^A / H_k^ψ spectrum power law

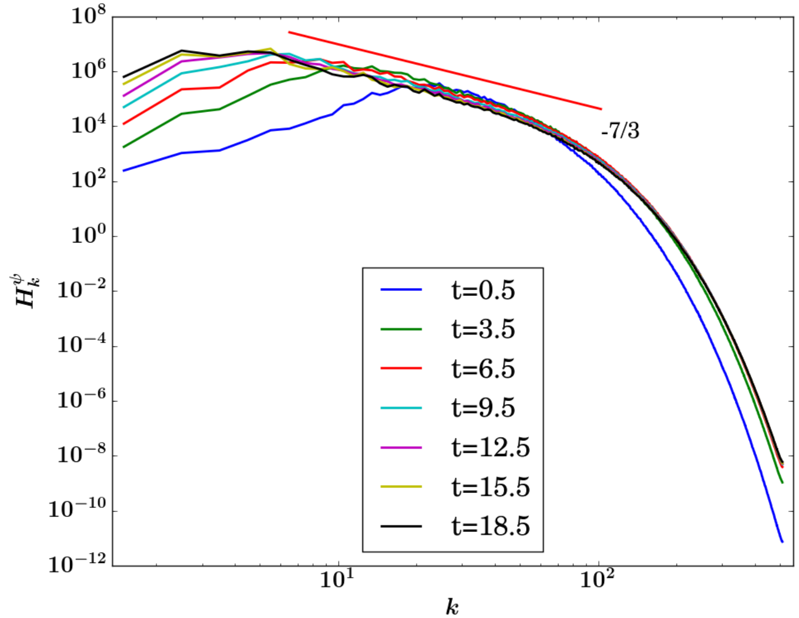
MHD



$$H_k^A \sim \epsilon_{HA}^{2/3} k^{-7/3}$$

Inverse cascade of H^A

CHNS



$$H_k^\psi \sim \epsilon_{H\psi}^{2/3} k^{-7/3}$$

Inverse cascade of H^ψ

The −7/3 power is robust. It does not change with the magnitude of external forcing

H_k^A / H_k^ψ spectrum power law

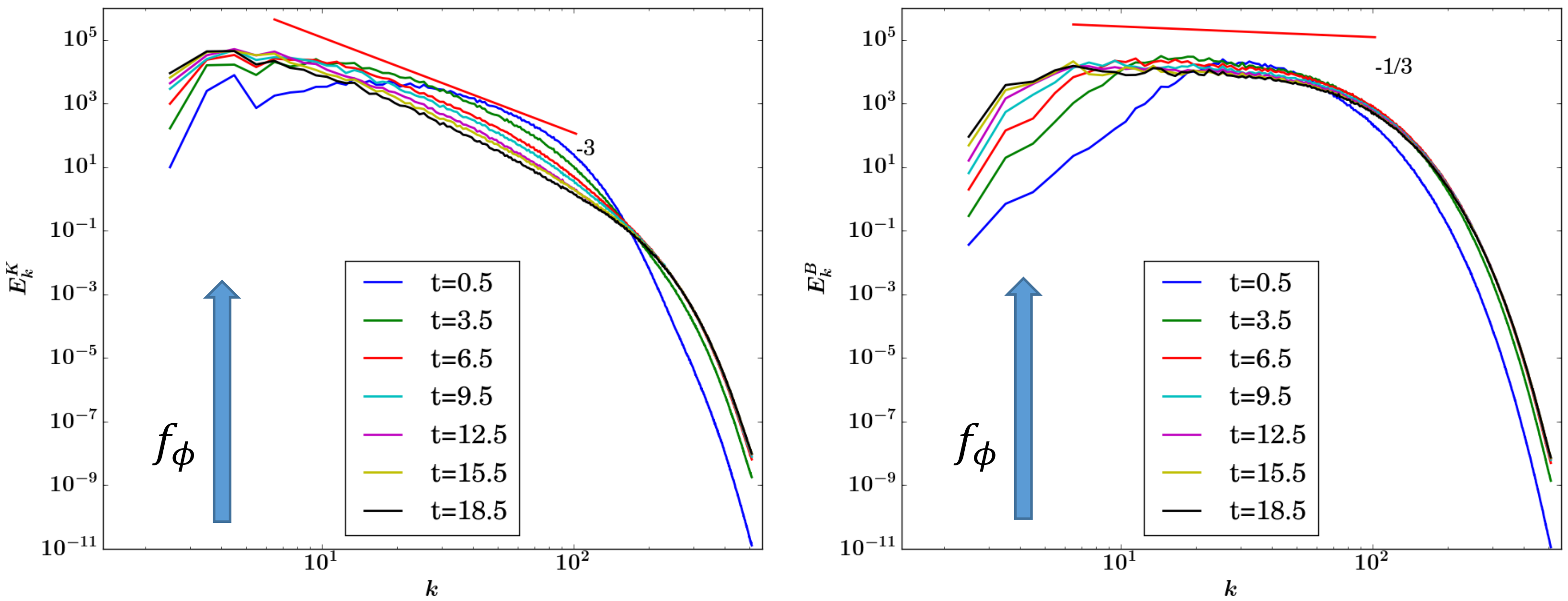
- Assuming a constant mean square magnetic potential dissipation rate ϵ_{HA}, according to the Alfvenic equipartition ($\rho\langle v^2\rangle \sim \frac{1}{\mu_0}\langle B^2\rangle$), the time scale for the decay of H^A ($\epsilon_{HA} \sim H^A/\tau$) can be estimated by $\tau \sim (vk)^{-1} \sim (Bk)^{-1}$.
- Define the spectrum to be $H^A = \sum_k H_k^A \sim kH_k^A$, so $B \sim kA \sim k(H^A)^{\frac{1}{2}} \sim (H_k^A)^{\frac{1}{2}} k^{\frac{3}{2}}$. Therefore $\epsilon_{HA} \sim \frac{H^A}{\tau} \sim (H_k^A)^{\frac{2}{3}} k^{\frac{7}{2}}$.

$$\boxed{H_k^A \sim \epsilon_{HA}^{2/3} k^{-7/3}}$$

- Similarly, we can obtain the H_k^ψ spectrum by assuming the elastic equipartition ($\rho\langle v^2\rangle \sim \xi^2\langle B_\psi^2\rangle$):

$$\boxed{H_k^\psi \sim \epsilon_{H\psi}^{2/3} k^{-7/3}}$$

Energy Spectrum Power Law

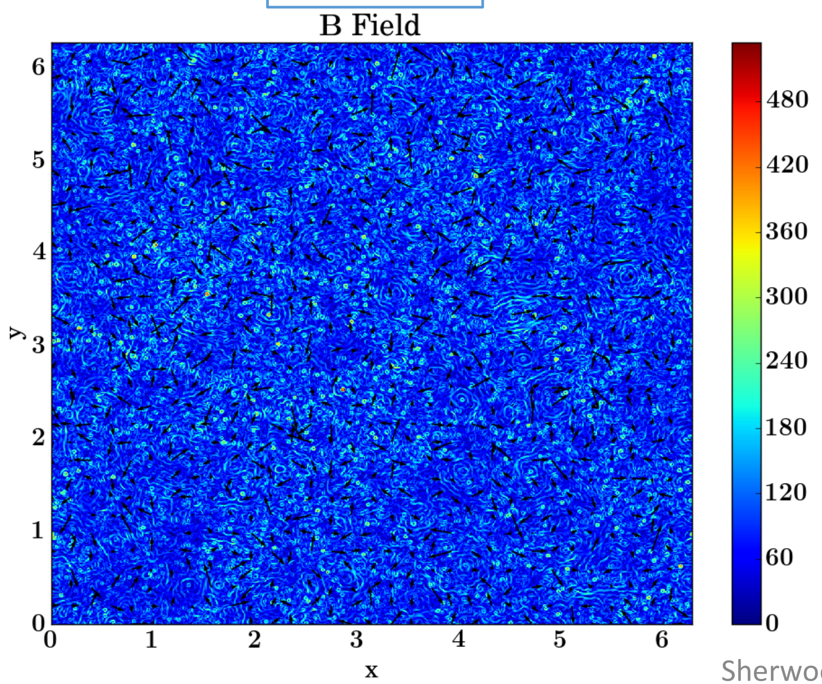


- $E_k^K \sim k^{-3}$ in 2D CHNS turbulence; on the other hand, it is well known that $E_k^K \sim k^{-3/2}$ in MHD turbulence.
- The -3 power law is consistent with the direct enstrophy cascade in 2D NS turbulence. The -3/2 power law comes from the Alfven effect (IK Theory).
- $H_k^\psi \sim k^{-7/3}$ vs. $H_k^A \sim k^{-7/3}$, the powers are the same. $E_k^K \sim k^{-3}$ for 2D CHNS; $E_k^K \sim k^{-3/2}$ for 2D MHD. ***Why?*** Why not -3/2?

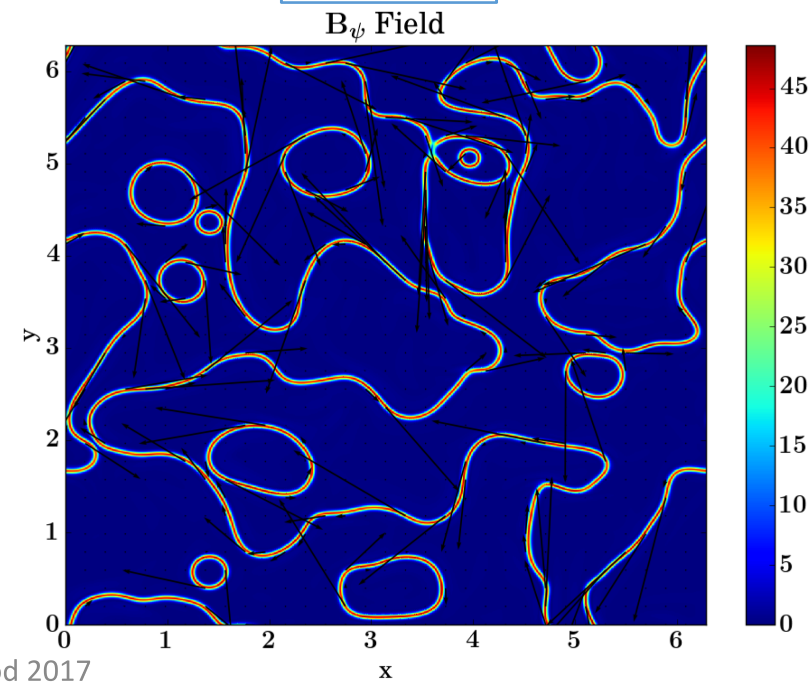
Interface Packing Matters!

- This initially surprising result is plausible because in the 2D CHNS system, B_ψ vanishes in most regions. Back reaction is apparently limited.
- On the other hand, the magnetic fields in MHD are not localized at specific regions, and Alfven waves can be everywhere.

MHD



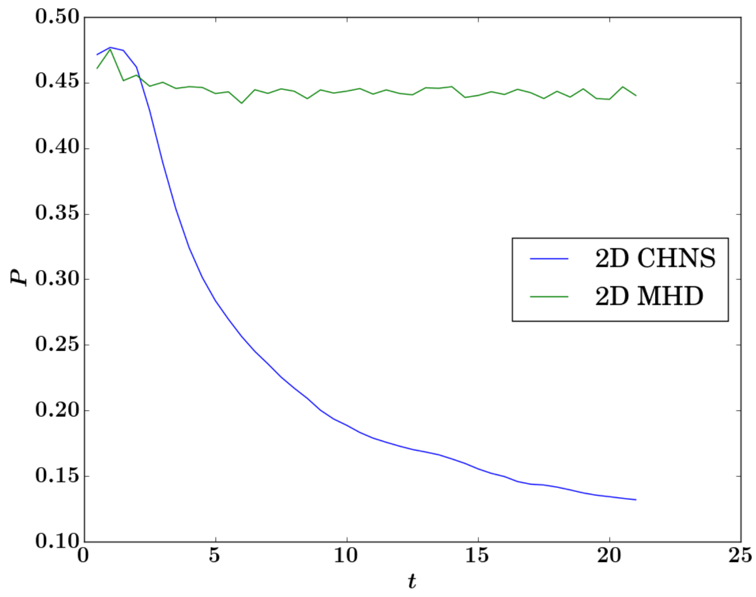
CHNS



Interface Packing Matters!

- Define the interface packing fraction P to be the ratio of mesh grid number where $|\vec{B}_\psi| > B_\psi^{rms}$ (or $|\vec{B}| > B^{rms}$) to the total mesh grid number.

$$\partial_t \omega + \vec{v} \cdot \nabla \omega = \frac{\xi^2}{\rho} \vec{B}_\psi \cdot \nabla \nabla^2 \psi + \nu \nabla^2 \omega$$

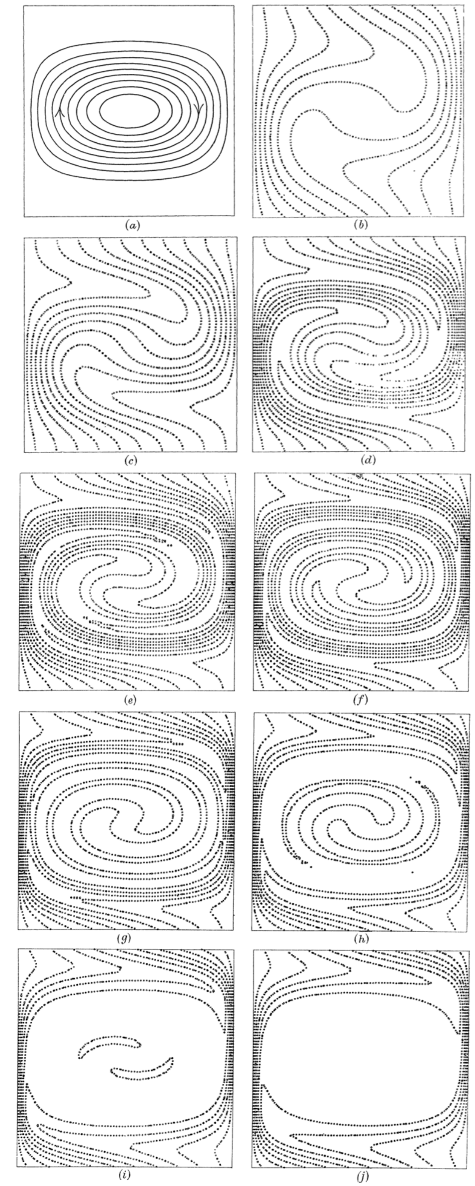


Turbulent Study: Results & Conclusions

- The study of elastic turbulence gives useful lessons.
- The H_k^{ψ} spectrum (inverse cascade of mean square blob density) is robust. It is an analogue to H_k^A spectrum (inverse cascade of H^A) in 2D MHD. On the other hand, the energy spectrum is NOT so robust.
- Interface Packing Matters! The spatial representation - i.e. tracking contours of bubble surfaces – can be a more revealing way of representing the turbulence than the traditional power law spectrum. PDF evolution should also be pursued.

Single Eddy Study: Flux Expulsion and Beyond

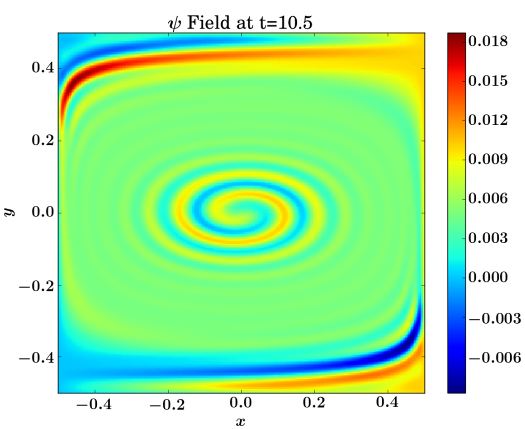
- When a convection eddy is imposed in a weak magnetic field, the magnetic field is expelled and amplified outside the eddy. This is called flux expulsion.
- Main results of Weiss 1966:
  - The final value of <B^2> can be estimated by $\langle B^2 \rangle \sim Rm^{1/2} B_0^2$
  - The time for <B^2> to reach a steady state is $\tau \sim Rm^{1/3} \tau_0$



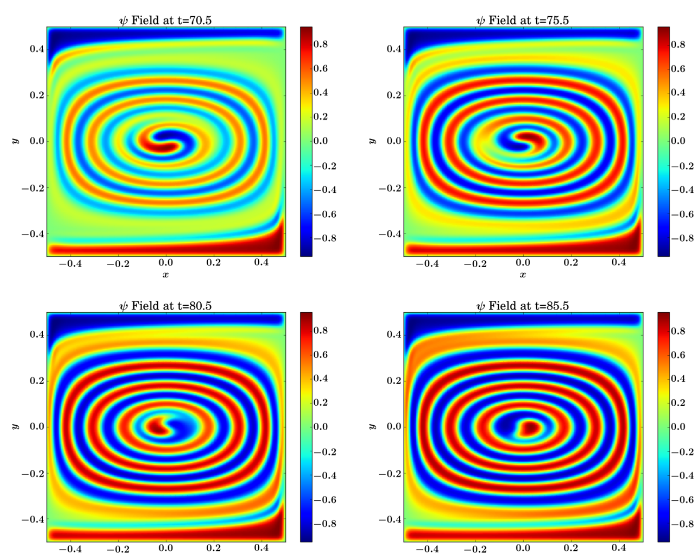
Weiss 1966

Single Eddy Study: Flux Expulsion and Beyond

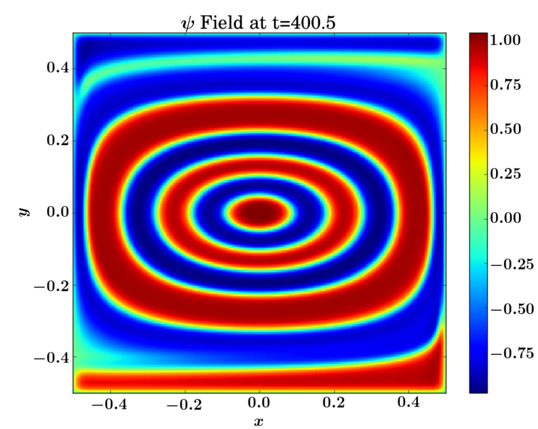
- To better understand the turbulence physics, we examine the evolution of the scalar concentration in a *single eddy* in the Cahn-Hilliard system. This extends the classic problem of flux expulsion in 2D MHD.
- There are 3 stages: the *"jelly roll"* stage, the *reconnection* stage, and the *target pattern* stage.



A. The jelly roll stage



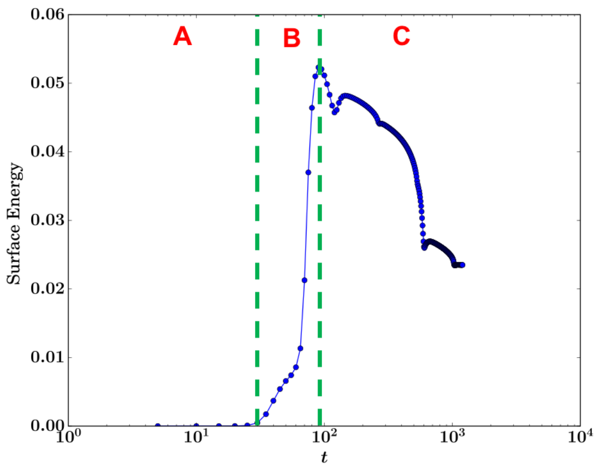
B. The reconnection stage

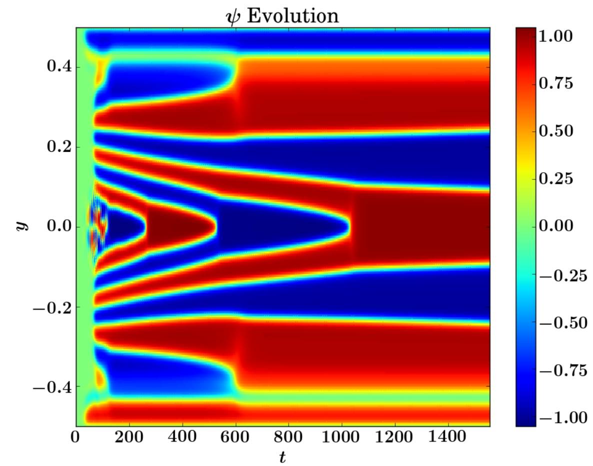


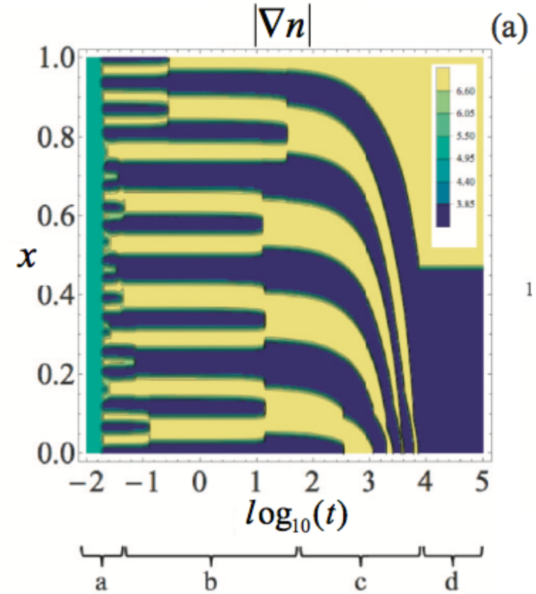
C. The target pattern stage

Single Eddy Study: Flux Expulsion and Beyond

- The bands merge on a time scale much longer than eddy turnover time.
- The 3 stages are reflected in the elastic energy plot.
- The target bands mergers are related to the dips in the target pattern stage.
- The band merger process is similar to the step merger in drift-ZF staircases.







Ashourvan *et.al.* 2016

Future Plans

- Further computations on
  - power law studies
  - forcing sensitivity
- Closure Model
  - Calculation of transport
- PV gradient? -> competing decompositions