

Basics of Turbulence I: A Look at Homogeneous Systems

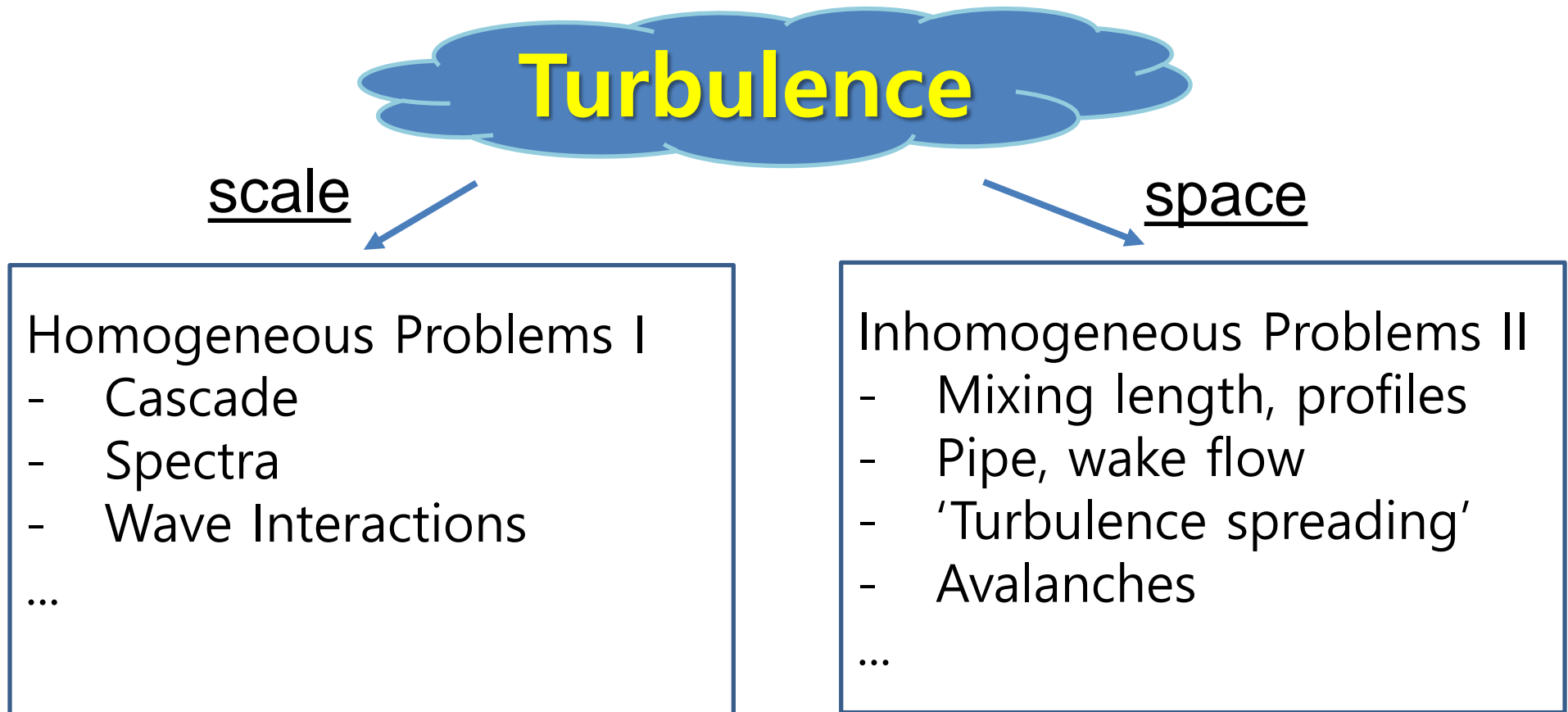
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Approach

- Highly Pedagogic



- Focus on simplest problems

Outline

- Basic Ideas
- K41 and Beyond
- Turbulence in Flatland – 2D Fluid Turbulence
- First Look at MHD Turbulence

Model

- Unless otherwise noted:

$$\rho \left(\frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla \vec{v} - \nu \nabla^2 \vec{v} \right) = -\nabla P + \tilde{f}$$

$$\nabla \cdot \vec{v} = 0$$

Random forcing
(usually large scale)

- Finite domain, closed, periodic
 - $Re = v \cdot \nabla v / \nu \nabla^2 v \sim VL/\nu$; $Re \gg 1$
- Variants:
 - 2D, QG
 - Compressible flow
 - Pipe flow – inhomogeneity

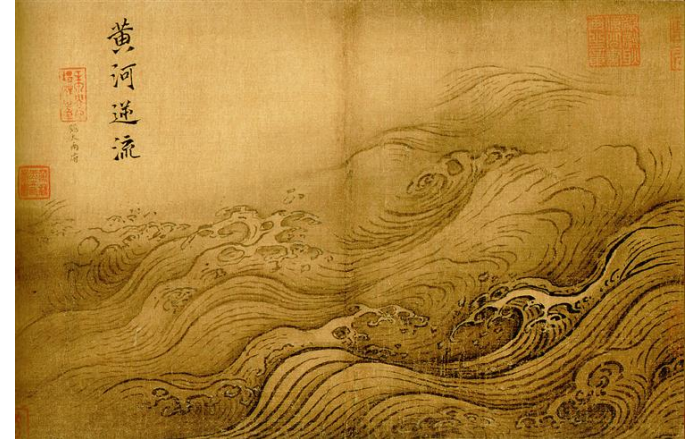
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What is turbulence?

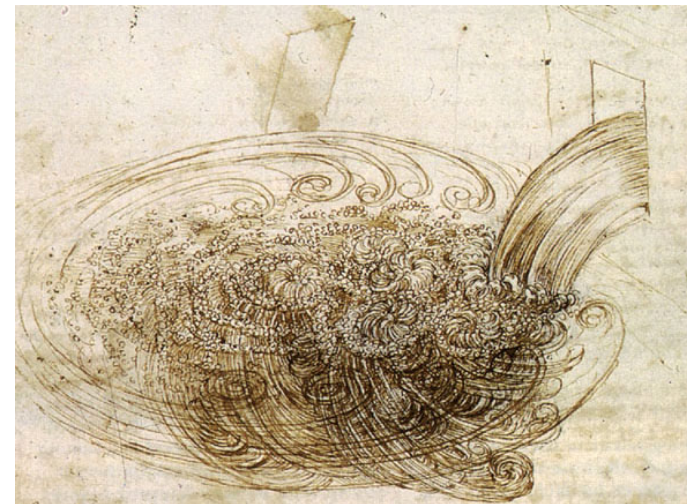
- Spatio-temporal “disorder”
- Broad range of space-time scales
- Power transfer / flux thru broad range of scales *
- Energy dissipation and irreversibility as $Re \rightarrow \infty$ *

And:

- Decay of large scales
- Irreversible mixing
- Intermittency / burstiness



Ma Yuan



Leonardo

What is difference between turbulence and noise/equilibrium fluctuations?

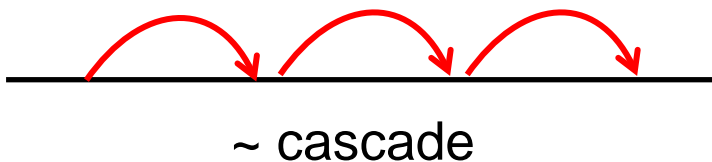
- Power transfer dominant
- Irreversibility for $\nu \rightarrow 0$
- Noisy thermal equilibrium: (ala' Test Particle Model)

Emission \leftrightarrow absorption balance, locally



Fluctuation-Dissipation Theorem applies

- Turbulence:



Flux \sim emission – absorption

Flux dominant for most scales

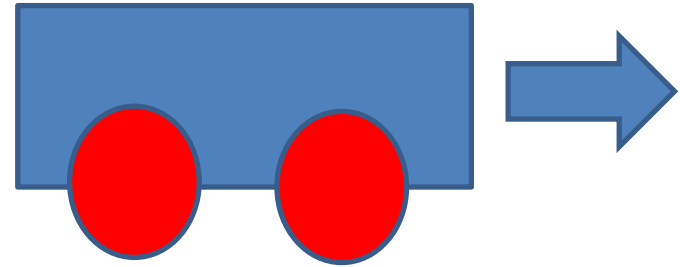
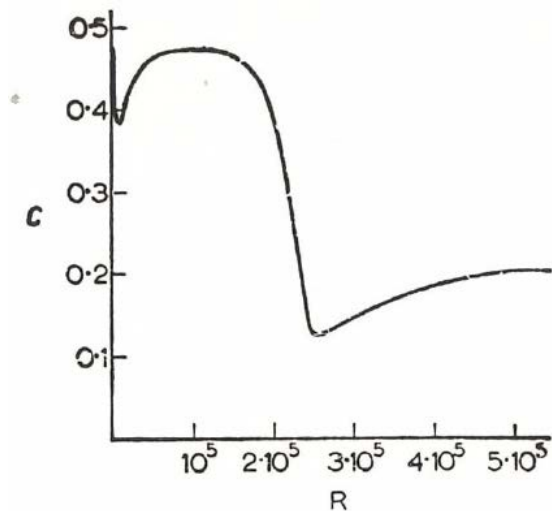
Why broad range scales?

What motivates cascade concept?

A) Planes, trains, automobiles...

DRAG

- Recall: $F_d \sim c_D \rho A V^2$
- $C_D = C_D(Re) \rightarrow$ drag coefficient



- The Point:

- Energy dissipation is finite, and due to viscosity, yet does not depend explicitly on viscosity → ANOMALY
- ‘Irreversibility persists as symmetry breaking factors vanish’

i.e. $\frac{dE}{dt} \sim F_d V \sim C_D \rho A V^3$

$$\frac{dE}{dt} \sim \frac{V^3}{l_0} \equiv \epsilon \rightarrow \text{dissipation rate} \quad l_0 \rightarrow \text{macro length scale}$$

- Where does the energy go?

$$\text{Steady state } \nu \langle (\nabla \vec{v})^2 \rangle = \langle \vec{f} \cdot \vec{v} \rangle = \epsilon$$

- So $\epsilon = \nu \langle (\nabla v)^2 \rangle \leftarrow$ independent of ν
 - $(\nabla v)_{rms} \sim \frac{1}{\nu^{1/2}} \rightarrow$ suggests \rightarrow singular velocity gradients (small scale)
- ∴
- Flat C_D in $Re \rightarrow$ turbulence must access small scales as $Re \rightarrow \infty$
 - Obviously consistent with broad spectrum, via nonlinear coupling

B) ... and balloons

- Study of ‘test particles’ in turbulence:
- Anecdotal:

Titus Lucretius Caro: 99-55 BC

“De rerum Nature” cf. section V, line 500

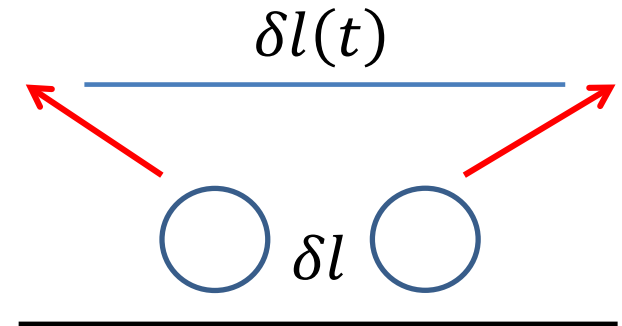
- Systematic:

L.F. Richardson: - probed atmospheric turbulence by study of balloon separation

Noted: $\langle \delta l^2 \rangle \sim t^3 \rightarrow$ super-diffusive

- not $\sim t$, ala’ diffusion, noise

- not exponential, ala’ smooth chaotic flow



Upshot:

$$\delta V(l) = \left(\left(\vec{v}(\vec{r} + \vec{l}) - \vec{v}(\vec{r}) \right) \cdot \frac{\vec{l}}{|\vec{l}|} \right) \rightarrow \text{structure function} \rightarrow \text{velocity differential across scale}$$

Then: $\delta V \sim l^\alpha$

so, $\frac{dl}{dt} \sim l^\alpha \rightarrow$ growth of separation

$\rightarrow \langle l^2 \rangle \sim t^{\frac{2}{1-\alpha}} \sim t^3$

$\rightarrow \alpha = \frac{1}{3}$

so $\delta V(l) \sim l^{1/3}, \langle \delta l^2 \rangle \sim t^3$

\rightarrow Points:

- large eddys have more energy, so rate of separation **increases** with scale
- **Relative** separation is excellent diagnostic of flow dynamics

cf: tetrads: Siggia and Shraiman

Roughness:

N.B. turbulence is spatially “rough”, i.e. $\delta V(l) \sim \epsilon^{1/3} l^{1/3}$

$$\lim_{l \rightarrow 0} \frac{V(\vec{r} + \vec{l}) - V(\vec{r})}{l} = \lim_{l \rightarrow 0} \frac{\delta V(l)}{l} = \epsilon^{1/3} / l^{2/3}$$

→ - strain rate increases on smaller scales

- turbulence develops progressively rougher structure on smaller scales

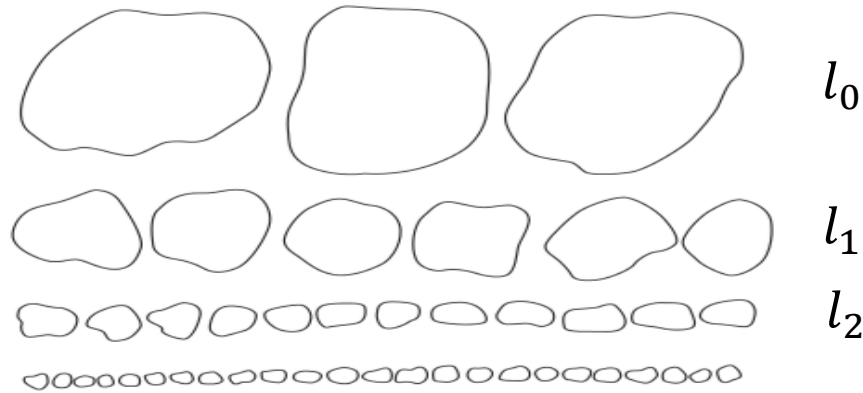
- Where are we?
 - turbulence develop singular gradients to maintain C_D indep. Re
 - turbulent flow structure exhibits
 - super-diffusive separation of test particles
 - power law scaling of $\delta V(l)$



- Cascade model – K41

K41 Model (Phenomenological)

- Cascade \rightarrow hierarchical fragmentation



- Broad range of scales, no gaps

- Described by structure function

- $\langle \delta V(l)^2 \rangle, \dots, \langle \delta V(l)^n \rangle, \dots$

\leftarrow Related to energy distribution
 \leftrightarrow greatest interest

- $\langle \delta v(l)^2 \rangle \leftrightarrow$ energy,
of great interest

- higher moments
more challenging

- Input:
- 2/3 law (empirical)

$$S_2(l) \sim l^{2/3}$$

- 4/5 law (Rigorous) - TBD

$$\langle \delta V(l)^3 \rangle = -\frac{4}{5} \epsilon l$$

→ Ideas:

- Flux of energy in scale space from l_0 (input/integral scale) to l_d (dissipation) scale – set by ν
- Energy flux is same at all scales between $l_0, l_d \leftrightarrow$ self-similarity

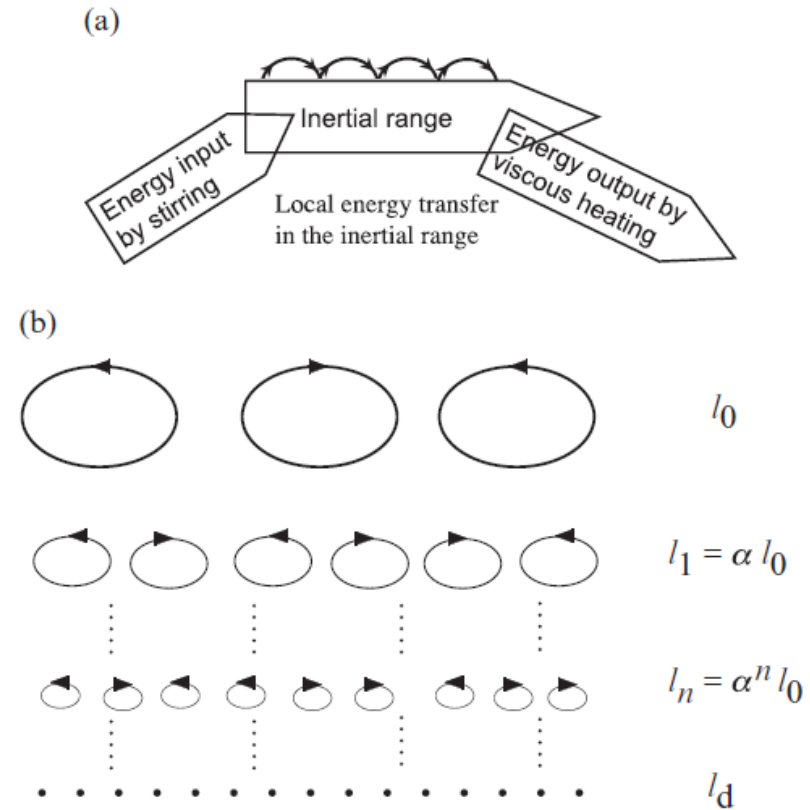
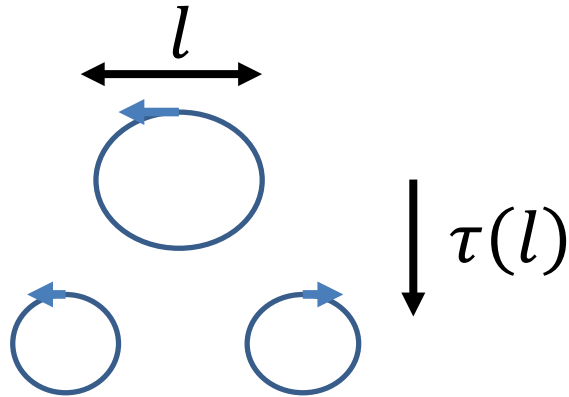


Fig. 2.12. Basic cartoon explanation of the Richardson–Kolmogorov cascade. Energy transfer in Fourier–space (a), and real scale (b)

And

- Energy dissipation – set as $\nu \rightarrow 0$ but not at $\nu = 0$
- * Asymmetry of breaking or stirring etc. lost in cascade: symmetry restoration
- N.B. intermittency \leftrightarrow ‘memory’ of stirring, etc
- Ingredients / Players
 - Exciton \rightarrow eddy (not a wave / eigenmode!)
 - l : scale parameter, eddy scale
 - $\delta V(l)$: velocity increment. Hereafter $V(l)$

- V_o : rms eddy fluctuation (large scale dominated)
- $\tau(l)$: \rightarrow eddy transfer / life-time / turn-over rate
- \rightarrow characteristic scale of transfer in cascade step



- Self-similarity \rightarrow constant flow-thru rate $\epsilon = V(l)^2 / \tau(l)$
- What is $\tau(l)$?? Consider...

The possibilities:

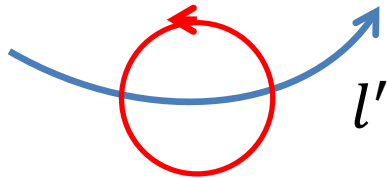
- Dimensionally, $\tau(l)$ is 'lifetime' of structure of scale l , time to distort out of existence

So

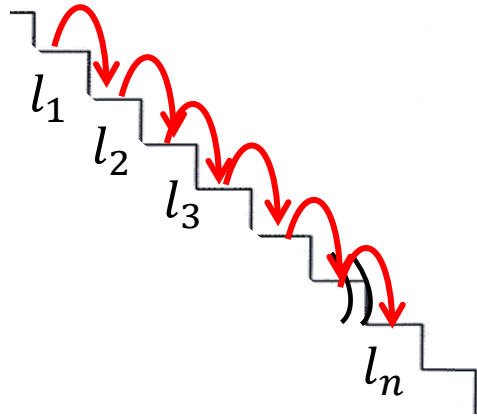
- $l' > l$
 - Larger scales advect eddy but don't distort it
 - Physics can't change under Galilean boostcf: Rapid distortions, shearing
- $l' < l$
 - Irrelevant \rightarrow insufficient energy



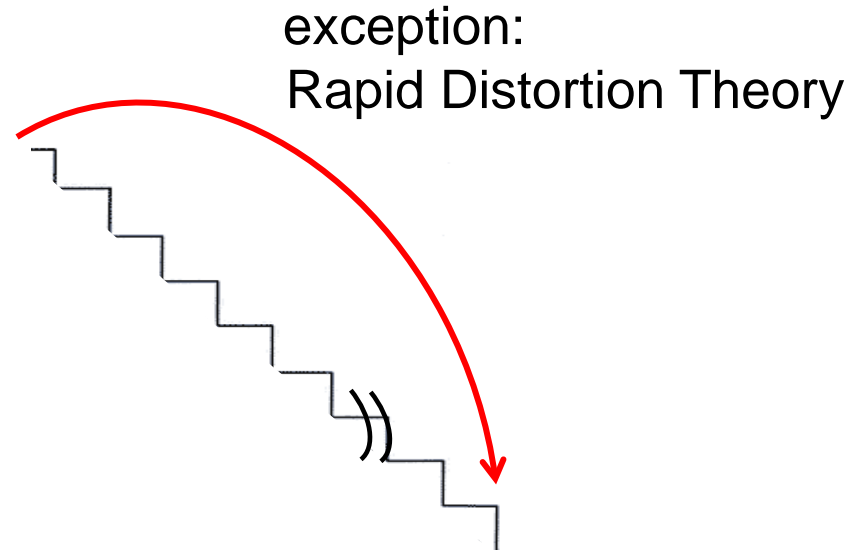
- $\tau(l) \sim l/V(l)$, set by $l' \sim l$



→ So



not



$$\rightarrow \epsilon \sim V(l)^2 / \tau(l) \sim V(l)^3 / l \rightarrow V(l) \sim (\epsilon l)^{1/3} ; 1 / \tau(l) \sim (\epsilon / l^2)^{1/3}$$

$$\rightarrow V(l)^2 \sim V_0^2 (l / l_0)^{2/3} \quad (\text{transfer rate increases as scale decreases})$$

And

$$\rightarrow E(k) \sim \epsilon^{2/3} k^{-5/3} \quad E = \int dk E(k)$$

→ Where does it end?

- Dissipation scale

- cut-off at $1/\tau(l) \sim \nu/l^2$ i.e. $Re(l) \rightarrow 1$

- $l_d \sim \nu^{3/4} / \epsilon^{1/4}$

- Degrees of freedom

$$\#DOFs \sim \left(\frac{l_o}{l_d}\right)^3 \sim Re^{9/4}$$

For $l_o \sim 1km$, $l_d \sim 1mm$ (PBL)

$\rightarrow N \sim 10^{18}$

→ Anything missing here?

- Dynamics!

- i.e.
- How is the energy transferred?
 - How are small scales generated?
 - Where have the N.S. equations gone?
 - ...

- Enter vorticity!

- $\omega = \nabla \times \vec{v}$; $\partial_t \vec{v} = \nabla \times \vec{v} \times \vec{\omega} + \nu \nabla^2 \vec{v}$

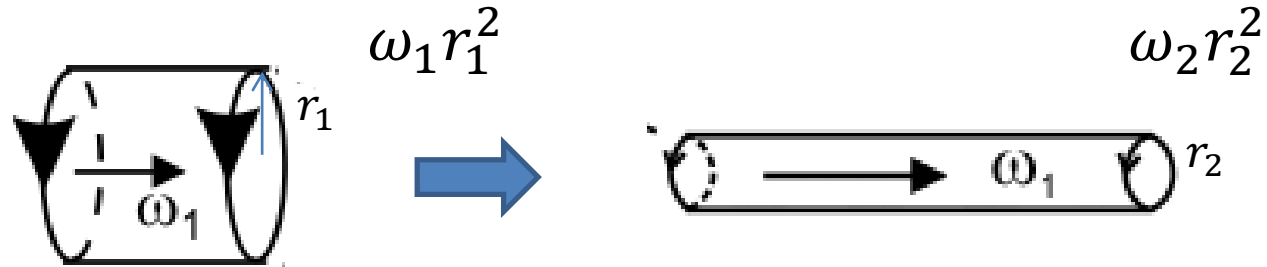
- $\Gamma = \int \oint \vec{v} \cdot d\vec{l} \sim \text{const. to } \nu$ (Kelvin's theorem)

So

- $\frac{\partial \vec{\omega}}{\partial t} + \vec{v} \cdot \nabla \vec{\omega} = \vec{\omega} \cdot \nabla \vec{v} + \nu \nabla^2 \vec{\omega}$ $\vec{\omega} \cdot \vec{S}$

Vortex tube stretching Strain tensor

- Stretching:



- Small scales generated ($\nabla \cdot \vec{v} = 0$)
- Energy transferred to small scale

- Enstrophy $\Omega = \langle \omega^2 \rangle$

$$\frac{d\omega^2}{dt} = \vec{\omega} \cdot (\vec{\omega} \cdot \nabla \vec{v}) + \dots \sim \omega^3 + \dots$$

- Enstrophy increases in 3D N-S turbulence
- Growth is strongly nonlinear

- Enstrophy production underpins forward energy cascade

- Where are we?

“Big whorls have little whorls that feed on their velocity. And little whorls have lesser whorls. An so on to viscosity.” – L.F. Richardson, 1920

Richardson, 1920

After: “So naturalists observe a flea has smaller fleas that on him prey; And these have smaller yet to bite 'em, And so proceed ad infinitum. Thus every poet, in his kind, Is bit by him that comes behind.” – Jonathan Swift, “On Poetry, a Rhapsody”, 1793

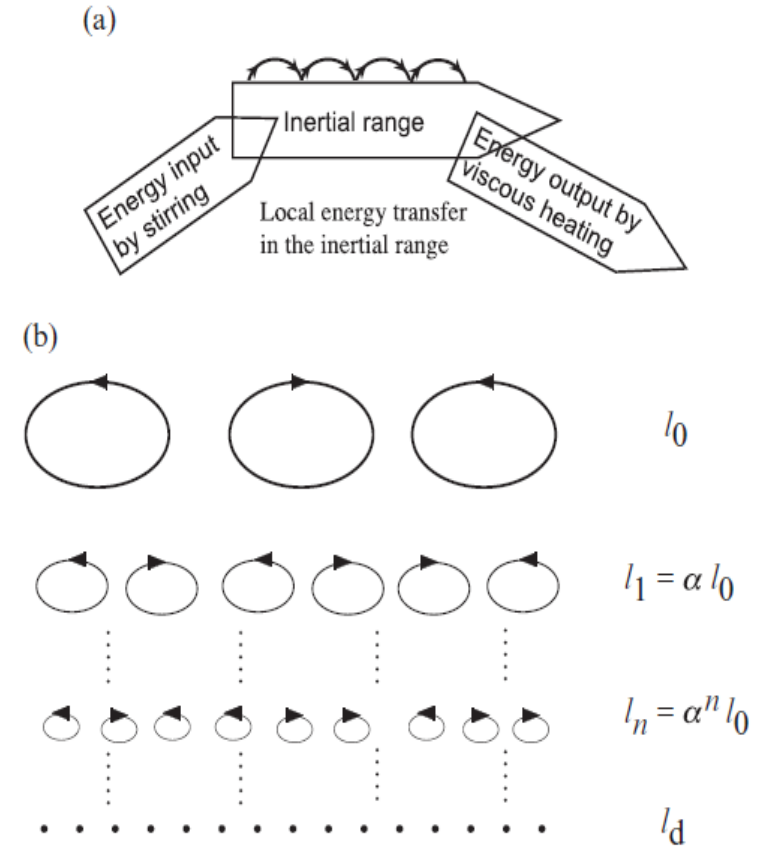


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The Theoretical Problem

- “We don’t want to *think* anything, man. We want to *know*.”
 - Marsellus Wallace, in “Pulp Fiction” (Quentin Tarantino)
- What do we know?
 - 4/5 Law (and not much else...)

$$\langle V(l)^3 \rangle = -\frac{4}{5} \epsilon l \rightarrow \text{asymptotic for finite } l, \nu \rightarrow 0$$
$$S_2 = \langle \delta V(l)^2 \rangle$$
$$S_3 = \langle \delta V(l)^3 \rangle$$

$$\text{from: } \frac{\partial S_2}{\partial t} = -\frac{1}{3l^4} \frac{\partial}{\partial l} (l^4 S_3) - \frac{4}{3} \epsilon + \frac{2\nu}{l^4} \frac{\partial}{\partial l} \left(l^4 \frac{\partial S_2}{\partial l} \right)$$

(Karman-Howarth)

flux in scale

dissipation

- Stationarity, $\nu \rightarrow 0$

4/5 Law

- $S_3(l) = -\frac{4}{5}\epsilon l$
- Energy thru-put balance $\langle \delta V(l)^3 \rangle / l \leftrightarrow \epsilon$
- Notable:
 - Euler: $\partial_t v + v \cdot \nabla v + \nabla P / \rho = 0$; reversible; $t \rightarrow -t, v \rightarrow -v$
 - N-S: $\partial_t v + v \cdot \nabla v + \nabla P / \rho = \nu \nabla^2 v$; time reversal broken by viscosity
 - $S_3(l)$: $S_3(l) = -\frac{4}{5}\epsilon l$; reversibility breaking maintained as $\nu \rightarrow 0$

Anomaly

- Extensions:

MHD: Pouquet, Politano

2D: Celari, et. al. (inverse cascade, only)

What of so called 'entropy cascade' in Vlasov turbulence?

- N.B.: A little history; philosophy:
 - ‘Anomaly’ in turbulence → Kolmogorov, 1941
 - Anomaly in QFT → J. Schwinger, 1951 (regularization for vacuum polarization)
- Speaking of QFT, what of renormalized perturbation theory?
 - Renormalization gives some success to low order moments, identifies relevant scales
 - Useful in complex problems (i.e. plasmas) and problems where τ_{int} is not obvious
 - Rather few fundamental insights have emerged from R.P.T

Caveat Emptor

Turbulence in Flat Land

- 2D systems \rightarrow 1 dimension constrained

i.e. Atmospheric \leftrightarrow rotation Ω_0

Magnetized plasma $\leftrightarrow \vec{B}_0, \Omega_c$

Solar interior \leftrightarrow stratification, ω_{B-V}

$$V/L \Omega_{eff} < 1$$

Low Rossby number

- Simple 2D fluid:

$$\frac{d\vec{\omega}}{dt} = \vec{\omega} \cdot \nabla \vec{v} + \nu \nabla^2 \vec{\omega}$$

$$\vec{v} = \nabla \phi \times \hat{z}$$

$$\omega = -\nabla^2 \phi$$



$$\partial_t \nabla^2 \phi + \nabla \phi \times \hat{z} \cdot \nabla \nabla^2 \phi = \nu \nabla^2 \nabla^2 \phi + \tilde{s}$$

forcing scale variable \rightarrow

- ω constant along fluid trajectories, to ν

- $\omega = \nabla^2 \phi$ akin conserved phase space density

$$\frac{\partial f}{\partial t} + v \frac{\partial f}{\partial x} + \frac{q}{m} E \frac{\partial f}{\partial v} = C(f)$$

- The problem:

- Enstrophy now conserved: $\vec{\omega} \cdot \nabla \vec{v} = 0$

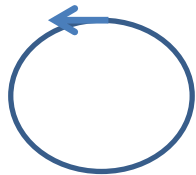
- Two inviscid invariants:

- Enstrophy $\Omega = \langle (\nabla^2 \phi)^2 \rangle$

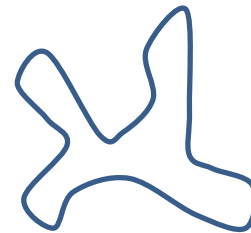
- Energy $E = \langle (\nabla \phi)^2 \rangle$

- Might ask: Where do these want to go, in scale?

- Enstrophy:



+ turbulent flow →



Isovorticity contour

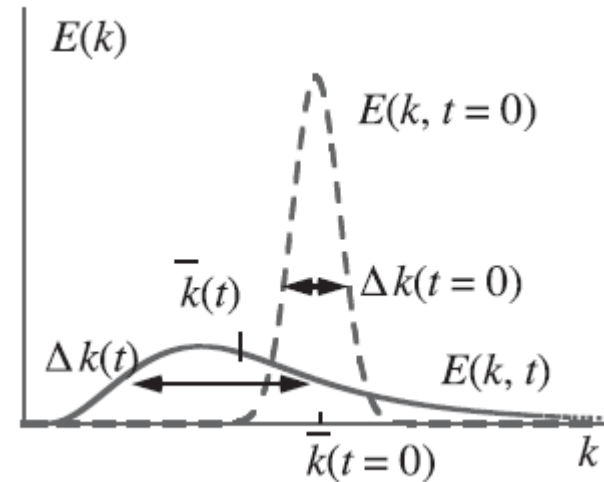
Stretched contour, $\langle (\nabla \omega)^2 \rangle \uparrow$
→ Enstrophy to small scale

- Energy

- Expect $(\Delta k)^2$ increases
- What of centroid \bar{k} ?

$$(\Delta k)^2 = \frac{1}{E} \int dk (k - \bar{k})^2 E(k)$$

$$\bar{k} = \frac{1}{E} \int dk E(k)$$



But

$$(\Delta k)^2 = \frac{1}{E} \int dk (k^2 - 2k\bar{k} + \bar{k}^2) E(k) = \frac{1}{\Omega} (\Omega - \bar{k}^2)$$

$$\partial_t (\Delta k)^2 > 0 \rightarrow \partial_t \bar{k} < 0$$

Ω conserved!

→ energy should head toward large scale

• Dilemma:

- Energy seeks large scale
- Enstrophy seeks small scale
- How accommodate self-similar transfer – i.e. cascade – of both?

➔ Dual cascade (R.H. Kraichnan)

- Forward self-similar transfer of enstrophy
 - ➔ toward small scale dissipation
- Inverse transfer of energy
 - ➔ scale independent dissipation?

(Low k sink)

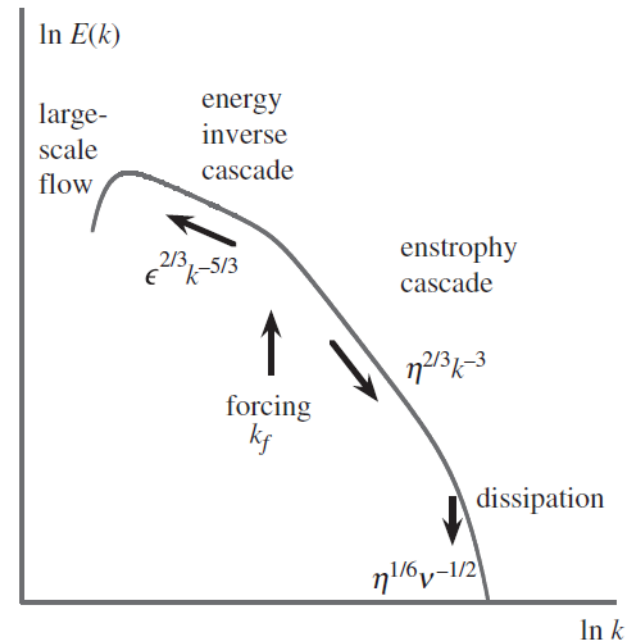


Fig. 2.17. Schematic of energy spectrum for dual cascade.

- Spectra

- Enstrophy range:

$$E(l) \rightarrow kE(k)$$

$$1/\tau(l) \rightarrow k[kE(k)]^{1/2}$$

$$\rightarrow E(k) = \eta^{2/3} k^{-3}$$

- Energy range: ala' K41; $E(k) = \epsilon^{2/3} k^{-5/3}$

- Pair dispersion:

- Energy range: ala' Richardson
- Enstrophy range: exponential divergence

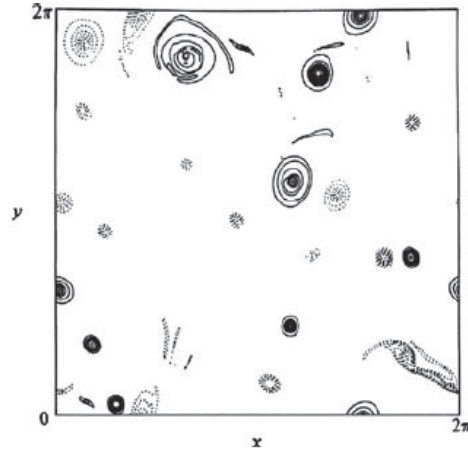
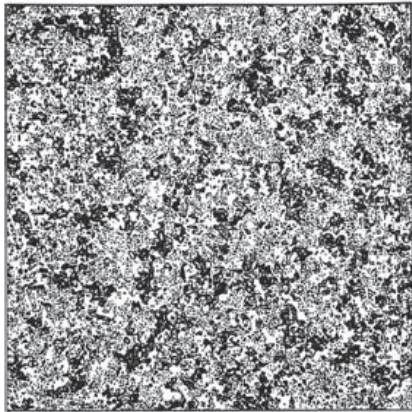
- Scale independent dissipation critical to stationary state

→ Where do we stand now?

“Big whorls meet bigger whorls, And so it tends to go on. By merging they grow bigger yet, And bigger yet, and so on.”

- M. McIntyre, after L.F. Richardson

- Cautionary tale: coherent structures happen!



Decay experiment

→ Isolated coherent vortices
appear in turbulent flow

McWilliams, '84 et. seq.

Herring and McWilliams '85

- Depending upon forcing, dynamics be cascade or coherent structure formation, or both:
- Need a non-statistical criterion, i.e. Okubo-Weiss

$$\rho = -\nabla^2 \phi, S = \partial^2 \phi / \partial x \partial y \rightarrow \text{local flow shear}$$

$$\partial_t \nabla \rho = (s^2 - \rho^2)^{1/2}; \text{ criterion for "coherence"}$$

→ Gaussian curvature of stream function predicts stability

- MHD turbulence - A First Look

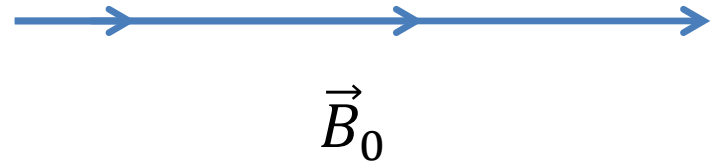
- HUGE subject – includes small scale and mean field dynamo problems (c.f. Hughes lectures)

- Here, focus on Alfvénic turbulence i.e. (Kraichnan-Iroshnikov-Goldreich-Sridhar ...) → wave turbulence

- Strong mean \vec{B}_0

- $\delta B < B_0, \nabla \cdot \vec{v} = 0$

- Shear-Alfvén wave turbulence



- Best described by reduced MHD: (Ohm's Law, $\nabla \cdot J = 0$)

$$\frac{\partial A_{\parallel}}{\partial t} + \nabla_{\perp} \phi \times \hat{z} \cdot \nabla_{\perp} A_{\parallel} = B_0 \partial_z \phi + \eta \nabla^2 A_{\parallel}$$

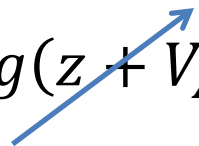
$$\frac{\partial}{\partial t} \nabla^2 \phi + \nabla_{\perp} \phi \times \hat{z} \cdot \nabla_{\perp} \nabla^2 \phi^2 = B_0 \partial_z \nabla^2 A_{\parallel} + \nabla_{\perp} A_{\parallel} \times \hat{z} \cdot \nabla_{\perp} \nabla^2 A_{\parallel} + \nu \nabla^2 \nabla^2 \phi + \tilde{S}$$

- Observations:
 - All nonlinear scattering is perpendicular
 - Contrast N-S, eddys with $\omega = 0$

Now: Alfvén waves: $\omega^2 = k_{\parallel}^2 V_A^2$

- If uni-directional wave population:

i.e. $A = f(z - V_A t) + g(z + V_A t)$



then f is exact solution of MHD

➔ Need counter-propagating populations to manifest nonlinear interaction

- See also resonance conditions

$$\begin{aligned} \omega_1 + \omega_2 &= \omega_3 \\ k_{\parallel 1} + k_{\parallel 2} &= k_{\parallel 3} \end{aligned}$$

- For Alfvén wave cascade:

$$\epsilon = T(k \rightarrow k + \Delta k) E(k) \rightarrow E(k) / \tau(k)$$


 transition rate

- Recall Fermi Golden Rule:

$$T_{i;j} \sim \frac{2\pi}{h} |\langle i | H_{int} | j \rangle|^2 \delta(E_j - E_i - h\omega)$$

$$\rightarrow T \sim \frac{V(l_d)^2}{l^2} \tau_{int}(l_{\perp})$$

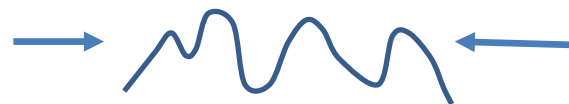
$V(l_{\perp})^2 \rightarrow$ scatter energy

$1/l^2 \rightarrow (cc)^2$

- $\tau_{int}(l) = 1/(\Delta k_{\parallel}) V_A$

\rightarrow Alfvénic transit time ($\Delta k_{\parallel} \sim k_{\parallel}$)

Packet passage



Enter the Kubo number

$$\frac{l_{\parallel ac}}{\Delta_{\perp}} \frac{\delta B}{B_0} \sim \left(\frac{V_A \delta B / B}{l_{\perp}} \right) |\Delta k_{\parallel} V_A|$$

- Basically: $B \cdot \nabla \rightarrow B_0 \partial_z + \tilde{B} \cdot \nabla_{\perp}$
 \rightarrow relative size $\left\{ \begin{array}{l} \text{Linear } B_0 \partial_z \\ \text{Nonlinear } \tilde{B} \cdot \nabla_{\perp} \end{array} \right.$

i.e. $K < 1 \rightarrow$ weak scattering, diffusion process

$K > 1 \rightarrow$ strong scattering, \sim de-magnetization \sim percolation

$K = 1 \rightarrow$ (critical) balance

Why Kubo?

- But... “It ain’t over till its over”

- Eastern (division) philosopher

- As l_{\perp} drops, $V(l_{\perp})/l_{\perp} \rightarrow (\Delta k_{\parallel})V_A$

→ $\tau_{\perp} \rightarrow \tau_{\parallel}$ $Ku \rightarrow 1$

- Critically balanced cascade, $Ku \sim 1$

i.e. $\frac{V(l_{\perp})}{l_{\perp}} \sim V_A \frac{\delta B(l_{\perp})}{B_0} \sim (\Delta k_{\parallel})V_A$, unavoidable at small scale

– Statement that transfer sets $K \approx 1$

$$k_{\parallel} = k_{\parallel}(l_{\perp})$$

– Attributed to G.-S. ‘95 but:

defines anisotropy

“the natural state of EM turbulence is $K \sim 1$ ”

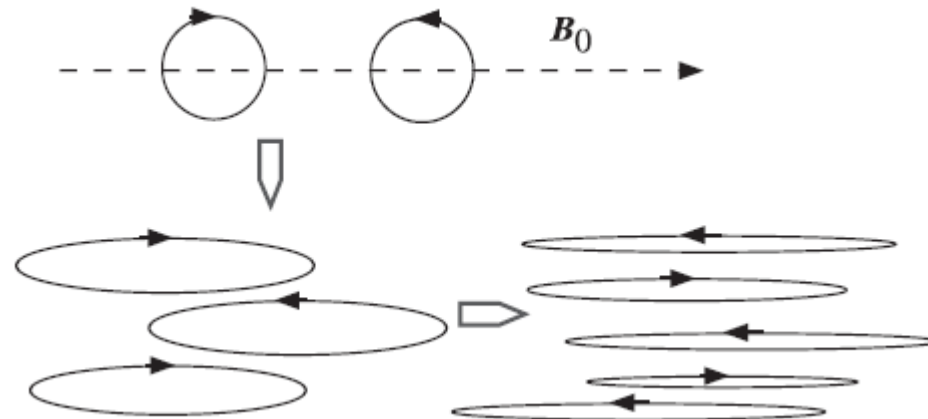
- Kadomtsev and Pogutse ‘78

- If now $\frac{1}{\tau_{int}(l_{\perp})} \sim \frac{V(l_{\perp})}{l_{\perp}}$

- Recover K41 scaling in MHD, $F(k_{\perp}) \sim \epsilon^{\frac{2}{3}} k_{\perp}^{-\frac{5}{3}}$

- “Great Power Law in the Sky”

- Eddy structure:



$$k_{\parallel} V_A \sim \frac{V(l_{\perp})}{l_{\perp}} \Rightarrow k_{\parallel} \sim k_{\perp}^{\frac{2}{3}} \epsilon^{\frac{1}{3}} / V_A \Rightarrow \text{anisotropy increases as } l_{\perp} \downarrow$$

- Many variants, extensions, comments, “we did it too’s”...

→ Fate of Energy?

- End point is dissipation
- What is dissipative structure?
 - Dimension < 3 → fractal and multi-fractal intermittency models
 - Structure:
 - Vortex sheet
 - Current sheet
 - Stability → micro-tearing, etc.
 - Energy leak to kinetic scales?
 - Electron vs ion heating
 - Particle acceleration (2nd order Fermi)

Conclusion

- This lecture is not even the “end of the beginning”
- A few major omissions:
 - pipe flow turbulence – Prandtl law of the wall
 - spatial structures, mixing, spreading
 - general theory of wave turbulence - Qiu, P.D.
 - MHDT + small scale dynamo - Hughes
 - kinetic/Vlasov turbulence – Sarazin, Qiu, Dif-Pradalier
 - Langmuir collapse ... - Kosuga