Turbulence and Transport in Elastic Systems: A Look at Some VERY Simple Examples

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- ➢Recent Collaboration:
  - Xiang Fan, Luis Chacon
- ➢Past Collaboration and Discussion:
  - D. W. Hughes, Steve Tobias, E. Kim, D. R. Nelson, F. Cattaneo, M. R. E. Proctor, A. Gruzinov, M. Vergassola, R. Pandit…

Outline

- **Models**

  -- What is an Elastic Fluid? (Pedagogic)

  - Oldroyd-B ‘family’, origins
  - MHD connection and Deborah number
  - Other systems, esp: Spinodal Decomposition in binary mixture

- **(Linked) Single Eddy**
  - Flux Expulsion – 2D MHD
    - Kinematics – two views
    - Dynamics – vortex disruption
  - Cahn-Hilliard Flows and Target Patterns

Outline

- **Turbulence**
  - 2D MHD – Quick Review
    - Dual cascade
    - A closer at $\langle \tilde{A}^2 \rangle$
  - Cahn-Hilliard Navier-Stokes (CHNS)
    - Scales, ranges, trends
    - Cascades and power laws
    - Lessons

Outline

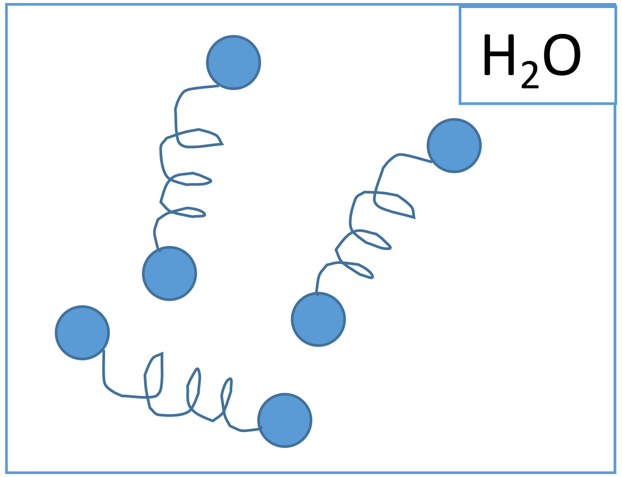
- **Active Scalar Transport**
  - 2D MHD – Flux Diffusion
    - Kinematics
    - Quenching: Alfvenization for vortex disruption
    - Thoughts on transport dynamics
  - CHNS -- ψ as the Active Scalar

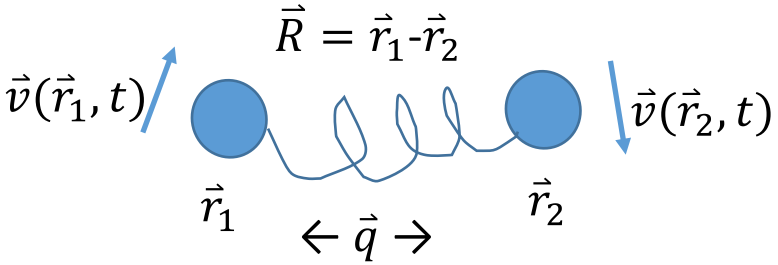
- **Conclusions, of Sorts**

Models

Elastic Fluid -> Oldroyd-B Family Models

→ Solution of Dumbells





Internal DoF
i.e. polymers

- $\gamma\left(\frac{d\vec{r}_{1,2}}{dt} - \vec{v}(\vec{r}_{1,2}, t)\right) = -\frac{\partial U}{\partial \vec{r}_{1,2}} + \vec{\xi}$, where $U = \frac{k}{2}(\vec{r}_1 - \vec{r}_2)^2 + \cdots$

stokes drag

entropic spring

noise

- so $\frac{d\vec{R}}{dt} = \vec{v}(\vec{R}, t) + \vec{\xi}/\gamma$, and $\frac{d\vec{q}}{dt} = \vec{q} \cdot \nabla\, \vec{v}(\vec{R}, t) - \frac{2}{\gamma}\frac{\partial U}{\partial \vec{q}}$ + noise

Seek $f(\vec{q}, \vec{R}, t|\vec{v}, \dots) \rightarrow$ distribution

- $\partial_t f + \partial_{\vec{R}} \cdot [\vec{v}(\vec{R},t)f] + \partial_{\vec{q}} \cdot \left[\vec{q} \cdot \nabla\vec{v}(\vec{R},t)f - \frac{2}{\gamma}\frac{\partial U}{\partial \vec{q}} f\right]$

$$= \partial_{\vec{R}} \cdot \mathbf{D}_0 \cdot \frac{\partial f}{\partial \vec{R}} + \partial_{\vec{q}} \cdot \mathbf{D}_\mathrm{q} \cdot \frac{\partial f}{\partial \vec{q}}$$

Is F.P. valid?

- and moments:

$Q_{ij}(\vec{R},t) = \int d^3q\, q_i q_j f(\vec{q},\vec{R},t)$ → electric energy field (tensor)

- so:

strain

$$\partial_t Q_{ij} + \vec{v} \cdot \nabla Q_{ij} = Q_{i\gamma}\partial_\gamma v_j + Q_{j\gamma}\partial_\gamma v_i$$
$$-\omega_z Q_{ij} + D_0 \nabla^2 Q_{ij} + 4\frac{k_B T}{\gamma}\delta_{ij}$$

relaxation

and concentration equation

- Defines Deborah number: $\nabla\vec{v}/\omega_z$

Reaction on Dynamics

- $\rho[\partial_t v_i + \vec{v} \cdot \nabla v_i] = -\nabla_i P + \nabla_i \cdot [\underset{\text{elastic stress}}{\underbrace{c_p k Q_{ij}}}] + \eta \nabla^2 v_i + f_i$
- Classic systems; Oldroyd-B (1950).
- Extend to nonlinear springs (FENE), rods, rods + springs, networks, director fields, etc…
- Supports elastic waves and fluid dynamics, depending on Deborah number.
- Oldroyd-B ↔ *active tensor* field

Constitutive Relations

- J. C. Maxwell:

$$(\text{stress}) + \overset{\substack{\text{relaxation}\\ \downarrow}}{\tau_R}\, \frac{d(\text{stress})}{dt} = \overset{\substack{\text{viscosity}\\ \downarrow}}{\eta}\, \frac{d}{dt}(\text{strain})$$

- If $\tau_R/T = D \ll 1$, stress = $\eta \frac{d}{dt}$(strain)

$$J = -\, \eta \nabla \vec{v}$$

- If $\tau_R/T = D \gg 1$, stress $\cong \frac{\eta}{\tau_R}$(strain)

$$\sim E\ (\text{strain})$$

- Limit of “freezing-in”: D>1 is criterion.

$T \equiv$ dynamic time scale

Relation to MHD?!

$\mathbf{T} \equiv$ stress

- Re-writing Oldroyd-B:

$$\frac{\partial}{\partial_t}\mathbf{T} + \vec{v}\cdot\nabla\mathbf{T} - \mathbf{T}\cdot\nabla\vec{v} - (\nabla\vec{v})^T\cdot\mathbf{T} = \frac{1}{\tau}(\mathbf{T} - \frac{\mu}{\tau}\mathbf{I})$$

- MHD: $\mathbf{T}_m = \frac{\vec{B}\vec{B}}{4\pi}$

$$\partial_t\vec{B} + \vec{v}\cdot\nabla\vec{B} = \vec{B}\cdot\nabla\vec{v} + \eta\nabla^2\vec{B}$$

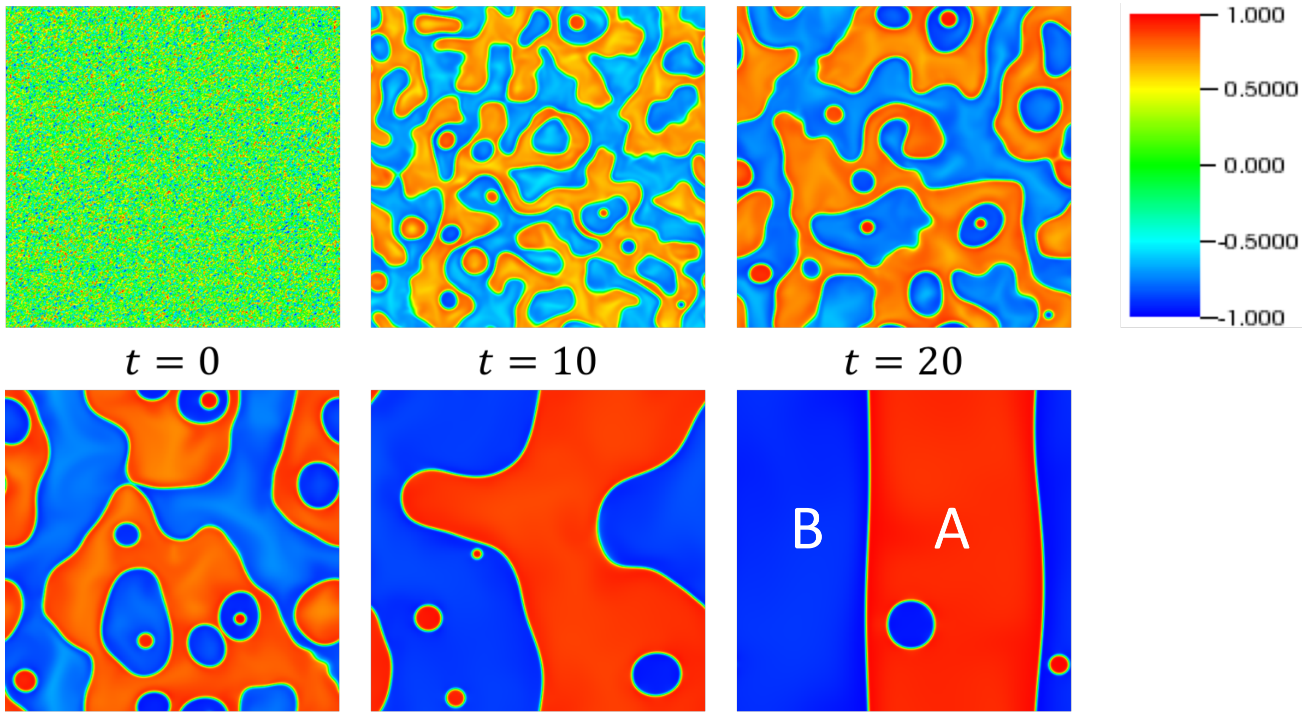
- So

$$\frac{\partial}{\partial_t}\mathbf{T}_m + \vec{v}\cdot\nabla\mathbf{T}_m - \mathbf{T}_m\cdot\nabla\vec{v} - (\nabla\vec{v})^T\cdot\mathbf{T}_m = \eta[\vec{B}\nabla^2\vec{B} + (\nabla^2\vec{B})\vec{B}]$$

- $\boxed{\lim_{D\to\infty}\text{(Oldroyd-B)} \Longleftrightarrow \lim_{R_m\to\infty}\text{(MHD)}}$

Elastic Media -- What Is the CHNS System

- ➢Elastic media – Fluid with internal DoFs → "springiness"
- ➢The Cahn-Hilliard Navier-Stokes (CHNS) system describes ***phase separation*** for binary fluid (i.e. ***Spinodal Decomposition***)



$t = 0$ $t = 10$ $t = 20$

$t = 40$ $t = 160$ $t = 350$

[Fan *et.al*. Phys. Rev. Fluids 2016]

Miscible phase
→ Immiscible phase

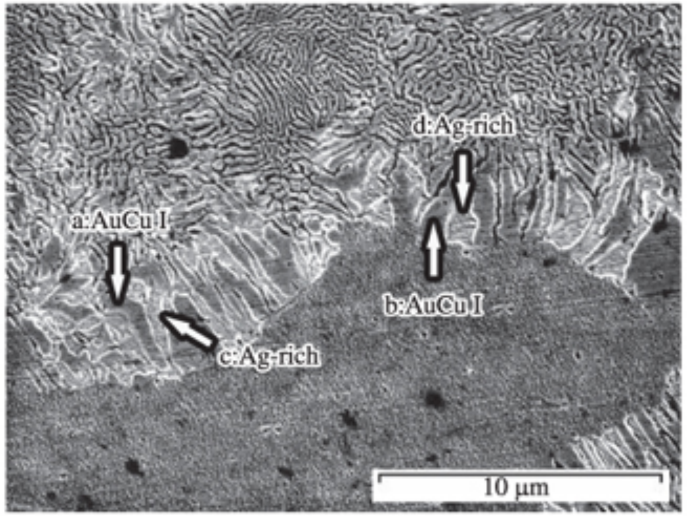


Figure 5. FE-SEM micrograph of specimen aged at 400 °C for 5000 minutes.

[Kim *et.al.* 2012]

Elastic Media? -- What Is the CHNS System?

- How to describe the system: the concentration field
- $\psi(\vec{r},t) \overset{\text{def}}{=} [\rho_A(\vec{r},t) - \rho_B(\vec{r},t)]/\rho$: scalar field → density contrast
- $\psi \in [-1,1]$
- CHNS equations (2D):

$$\boxed{\begin{aligned}\partial_t\psi + \vec{v}\cdot\nabla\psi &= D\nabla^2(-\psi + \psi^3 - \xi^2\nabla^2\psi)\\ \partial_t\omega + \vec{v}\cdot\nabla\omega &= \frac{\xi^2}{\rho}\vec{B}_\psi\cdot\nabla\nabla^2\psi + \nu\nabla^2\omega\end{aligned}}$$

Why Should a Plasma Physicist Care?

- Useful to examine familiar themes in plasma turbulence from new vantage point
- Some key issues in plasma turbulence:

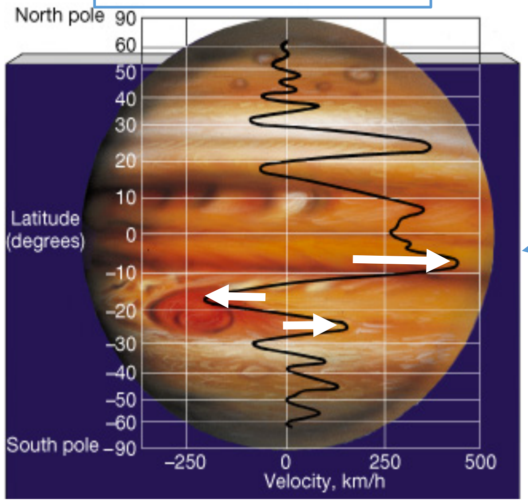
1. Electromagnetics Turbulence
   - CHNS vs 2D MHD: analogous, with interesting differences.
   - Both CHNS and 2D MHD are ***elastic*** systems
   - Most systems = 2D/Reduced MHD + many linear effects
     - Physics of dual cascades and constrained relaxation → relative importance, selective decay…
     - Physics of wave-eddy interaction effects on nonlinear transfer (i.e. Alfven effect ←→ Kraichnan)

MHD ←→ CHNS

Why Care?

2. Zonal flow formation → negative viscosity phenomena
    - ZF can be viewed as a "spinodal decomposition" of momentum.
    - What determines scale?

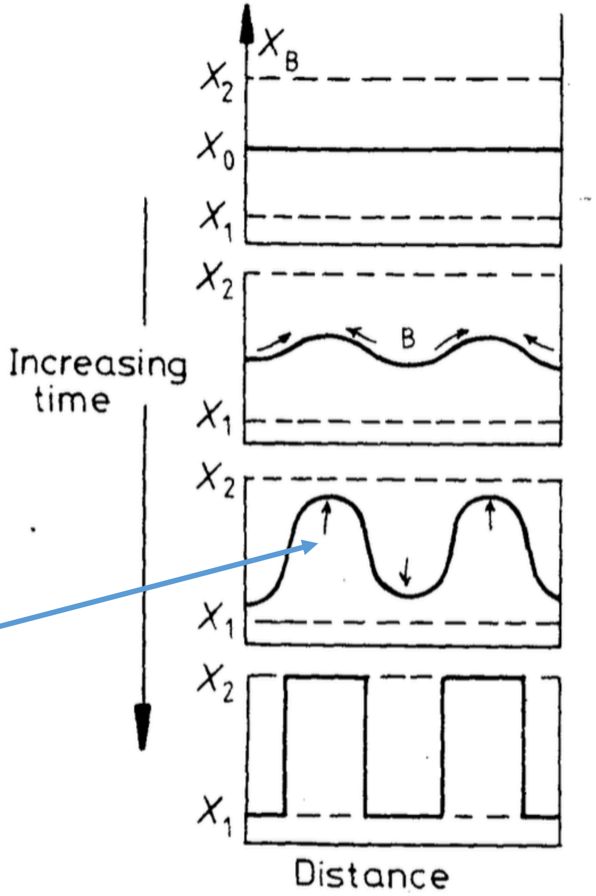
Zonal Flow



http://astronomy.nju.edu.cn/~lixd/GA/AT4/AT411/HTML/AT41102.htm

Arrows:
ψ for CHNS;
flow for ZF.

Spinodal Decomposition



[Porter 1981]

Why Care?

3. “Blobby Turbulence”
    - CHNS is a naturally blobby system of turbulence.
    - What is the role of structure in interaction?
    - How to understand blob coalescence and relation to cascades?
    - How to understand multiple cascades of blobs and energy?

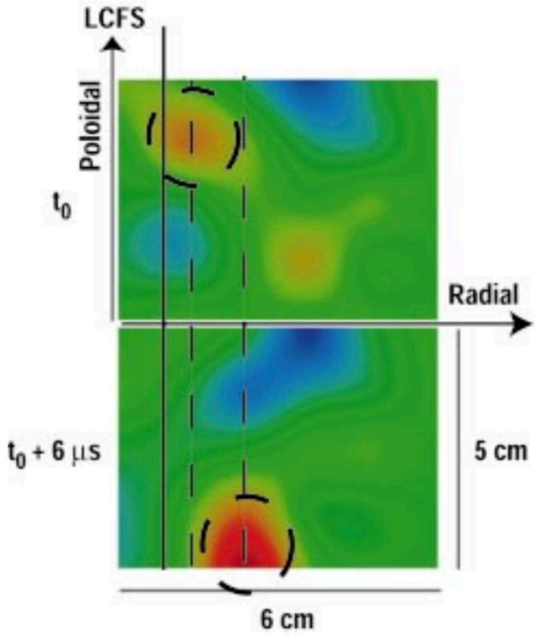


FIG. 4. (Color) Two frames from BES showing 2-D density plots. There is a time difference of 6 μs between frames. Red indicates high density and blue low density. A structure, marked with a dashed circle and shown in both frames, features poloidal and radial motion.

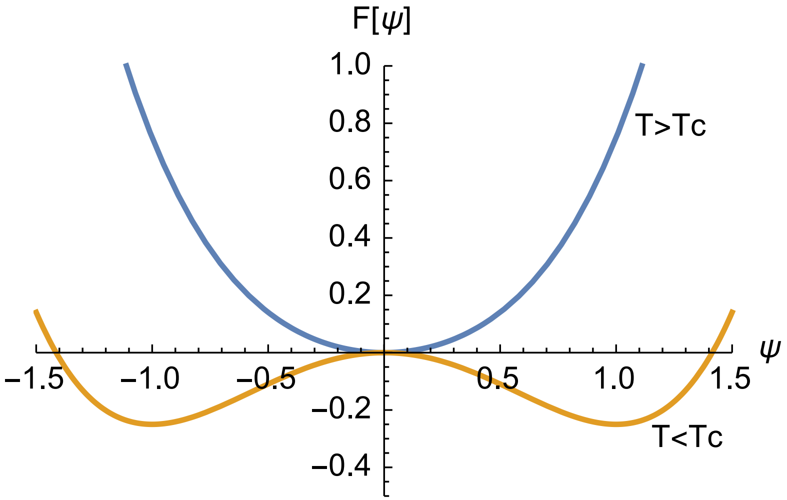
[J. A. Boedo et.al. 2003]

- CHNS exhibits all of the above, with many new twists

A Brief Derivation of the CHNS Model

- Second order phase transition → Landau Theory.
- Order parameter: $\psi(\vec{r},t) \overset{\text{def}}{=} [\rho_A(\vec{r},t) - \rho_B(\vec{r},t)]/\rho$
- Free energy:

$$\mathrm{F}(\psi) = \int d\vec{r}\Big(\underbrace{\frac{1}{2}C_1\psi^2 + \frac{1}{4}C_2\psi^4}_{\text{Phase Transition}} + \underbrace{\frac{\xi^2}{2}|\nabla\psi|^2}_{\text{Gradient Penalty}}\Big)$$



- $C_1(T), C_2(T)$.
- Isothermal $T < T_C$. Set $C_2 = -C_1 = 1$:

$$\mathrm{F}(\psi) = \int d\vec{r}\Big(-\frac{1}{2}\psi^2 + \frac{1}{4}\psi^4 + \frac{\xi^2}{2}|\nabla\psi|^2\Big)$$

A Brief Derivation of the CHNS Model

- Continuity equation: $\frac{d\psi}{dt} + \nabla \cdot \vec{J} = 0$. Fick's Law: $\vec{J} = -D\nabla\mu$.
- Chemical potential: $\mu = \frac{\delta F(\psi)}{\delta\psi} = -\psi + \psi^3 - \xi^2\nabla^2\psi$.
- Combining above → Cahn Hilliard equation:

$$\boxed{\frac{d\psi}{dt} = D\nabla^2\mu = D\nabla^2(-\psi + \psi^3 - \xi^2\nabla^2\psi)}$$

- $d_t = \partial_t + \vec{v} \cdot \nabla$. Surface tension: force in Navier-Stokes equation:

$$\boxed{\partial_t\vec{v} + \vec{v} \cdot \nabla\vec{v} = -\frac{\nabla p}{\rho} - \psi\nabla\mu + \nu\nabla^2\vec{v}}$$

- For incompressible fluid, $\nabla \cdot \vec{v} = 0$.

2D CHNS and 2D MHD

- 2D CHNS Equations:

$$\partial_t \psi + \vec{v} \cdot \nabla \psi = D \nabla^2 (-\psi + \psi^3 - \xi^2 \nabla^2 \psi)$$

$$\partial_t \omega + \vec{v} \cdot \nabla \omega = \frac{\xi^2}{\rho} \vec{B}_\psi \cdot \nabla \nabla^2 \psi + \nu \nabla^2 \omega$$

$-\psi$: Negative diffusion term

ψ^3: Self nonlinear term

$-\xi^2 \nabla^2 \psi$: Hyper-diffusion term

With $\vec{v}=\hat{\hat{z}} \times \nabla \phi$, $\omega = \nabla^2 \phi$, $\vec{B}_\psi = \hat{\hat{z}} \times \nabla \psi$, $j_\psi = \xi^2 \nabla^2 \psi$.

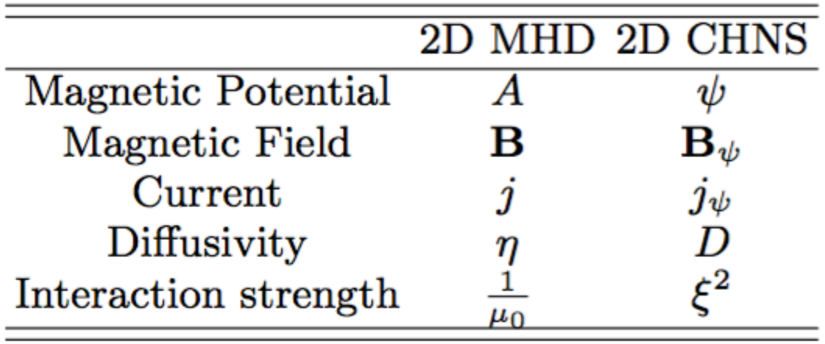
- 2D MHD Equations:

$$\partial_t A + \vec{v} \cdot \nabla A = \eta \nabla^2 A$$

$$\partial_t \omega + \vec{v} \cdot \nabla \omega = \frac{1}{\mu_0 \rho} \vec{B} \cdot \nabla \nabla^2 A + \nu \nabla^2 \omega$$

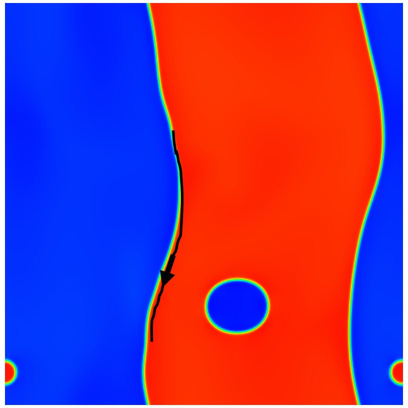
A: Simple diffusion term

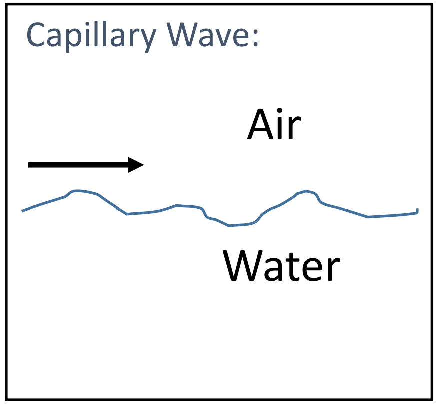
With $\vec{v}=\hat{\hat{z}} \times \nabla \phi$, $\omega = \nabla^2 \phi$, $\vec{B} = \hat{\hat{z}} \times \nabla A$, $j = \frac{1}{\mu_0} \nabla^2 A$.



	2D MHD	2D CHNS
Magnetic Potential	A	ψ
Magnetic Field	$\mathbf{B}$	$\mathbf{B}_\psi$
Current	j	j_ψ
Diffusivity	η	D
Interaction strength	$\frac{1}{\mu_0}$	ξ^2

Linear Wave





- CHNS supports linear “elastic” wave:

$$\omega(k) = \pm\sqrt{\frac{\xi^2}{\rho}}\left|\vec{k}\times\vec{B}_{\psi 0}\right| - \frac{1}{2}i(CD+\nu)k^2$$

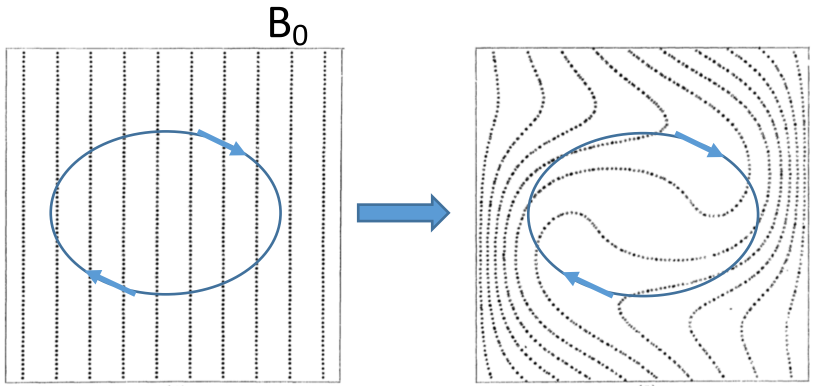
Where $C \equiv [-1 - 6\psi_0\nabla^2\psi_0/k^2 - 6(\nabla\psi_0)^2/k^2 - 6\psi_0\nabla\psi_0\cdot i\mathbf{k}/k^2 + 3\psi_0^2 + \xi^2k^2]$

- Akin to capillary wave at phase interface. Propagates ***only*** along the interface of the two fluids, where $|\vec{B}_\psi| = |\nabla\psi| \neq 0$.
- Analogue of Alfven wave.
- Important differences:
  - $\vec{B}_\psi$ in CHNS is large only in the interfacial regions.
  - Elastic wave activity does not fill space.

(Linked) Single Eddy

Flux Expulsion

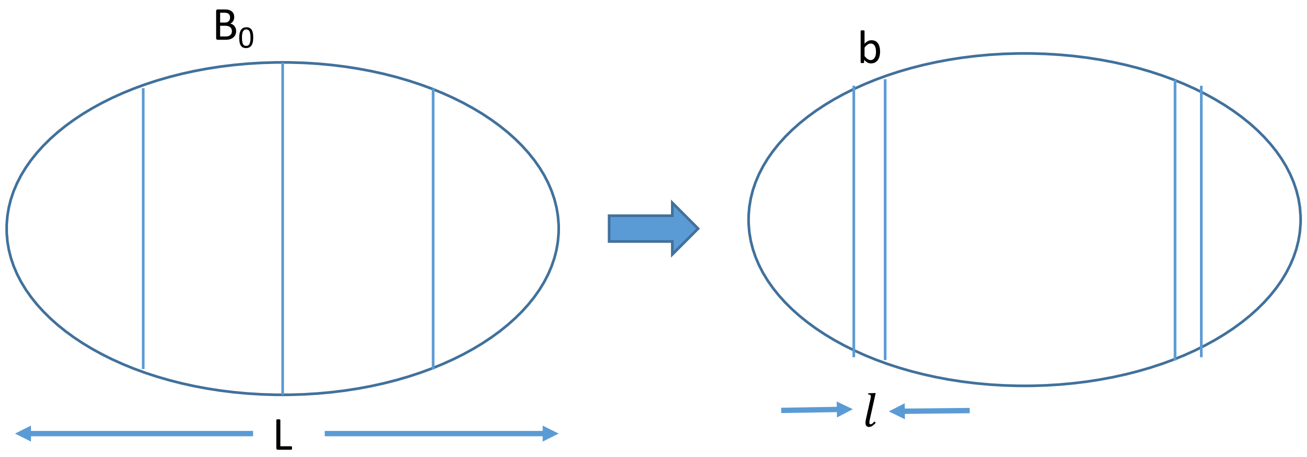
- Simplest dynamical problem in MHD (Weiss ‘66, et. seq.)
- Closely related to “PV Homogenization”



$Rm \sim vL/\eta \gg 1$

- Field wound-up, “expelled” from eddy
- For large Rm, field concentrated in boundary layer of eddy
- Ultimately, back-reaction asserts itself for sufficient B_0

How to Describe?



after n turns:
nl=L

- Flux conservation: $B_0L \sim bl$ Wind up: $b=nB_0$ (field stretched)
- Rate balance: wind-up ~ dissipation

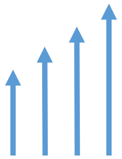
$$\frac{v}{L}B_0 \sim \frac{\eta}{l^2}b\ . \quad \tau_{expulsion} \sim \left(\frac{L}{v_0}\right)Rm^{1/3}.$$

$$l \sim \delta_{BL} \sim L/Rm^{1/3}\ . \quad b \sim Rm^{1/3}B_0\ .$$

N.B. differs from
Sweet-Parker!

What's the Physics?

- Shear dispersion! (Moffatt, Kamkar '82)



$$\partial_t A + \vec{v} \cdot \nabla A = \eta \nabla^2 A \quad \text{(Shearing coordinates)}$$

$$v_y = v_y(x) = v_{y0} + x v_y' + \cdots$$

$$\frac{dk_x}{dt} = -k_y v_y' \, , \frac{dk_y}{dt} = 0$$

$$\partial_t A + x v_y' \partial_y A - \eta\left(\partial_x^2 + \partial_y^2\right) A = 0$$

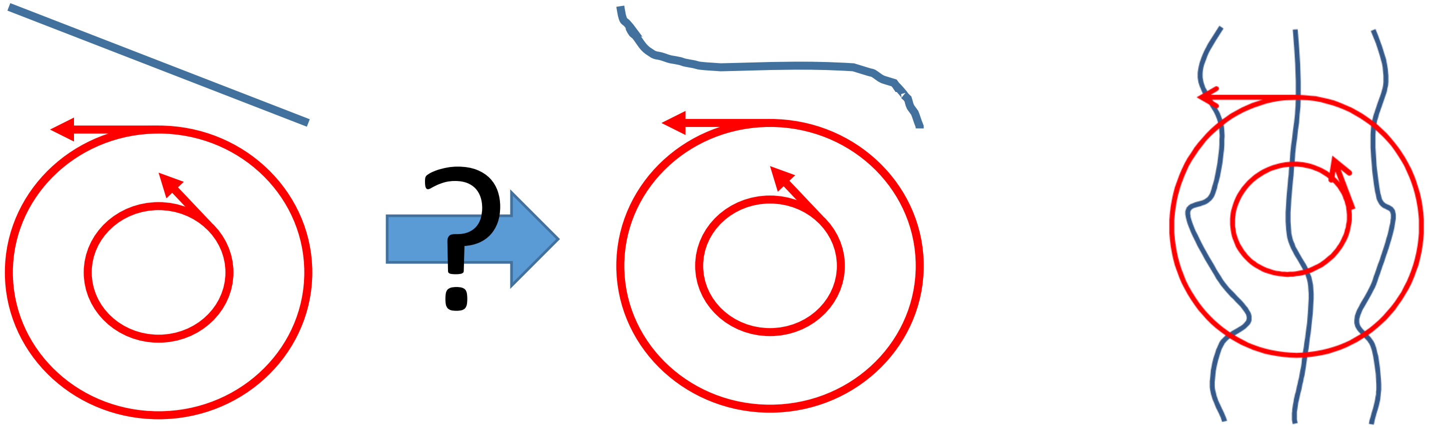
$$A = A(t) \exp i(\vec{k}(t) \cdot \vec{x})$$

(Shear enhanced dissipation annihilates interior field)

- So $\tau_{mix} \cong \tau_{shear} Rm^{1/3}$=$({v_y'}^{-1})Rm^{1/3}$

Single Eddy Mixing -- Cahn-Hilliard

- Structures are the key → need understand how a ***<u>single eddy</u>*** interacts with ψ field
- Mixing of $\nabla\psi$ by a single eddy → characteristic time scales?
- Evolution of structure?
- Analogous to flux expulsion in MHD (Weiss, '66)



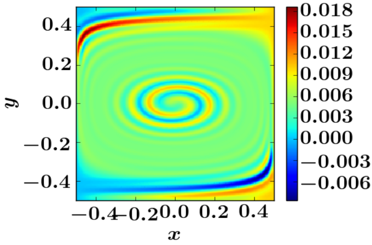
$\nabla\psi \leftrightarrow \vec{B}$

Transport / Relaxation

Single Eddy Mixing -- Cahn-Hilliard

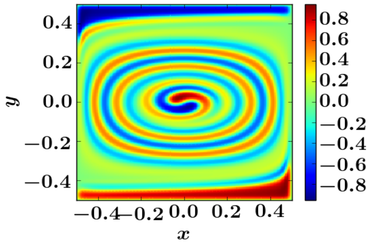
- ➢3 stages: (A) the *"jelly roll"* stage, (B) the *topological evolution* stage, and (C) the *target pattern* stage.
- ➢ψ ultimately homogenized in slow time scale, but metastable target patterns formed and merge.

A: Jelly roll

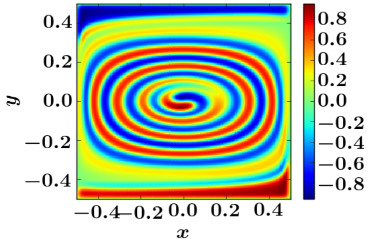


(a) t=10

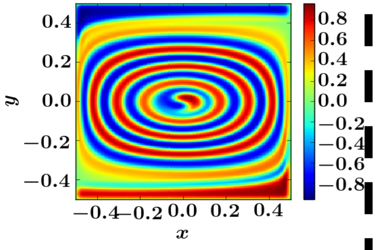
B: reconnection



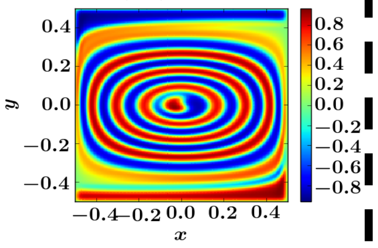
(b) t=70



(c) t=75

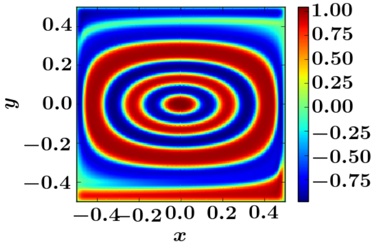


(d) t=80

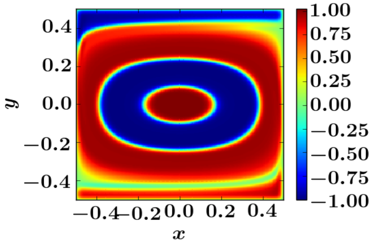


(e) t=85

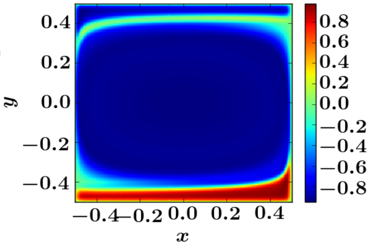
C: Target



(f) t=400



(g) t=1500



(h) t=4000

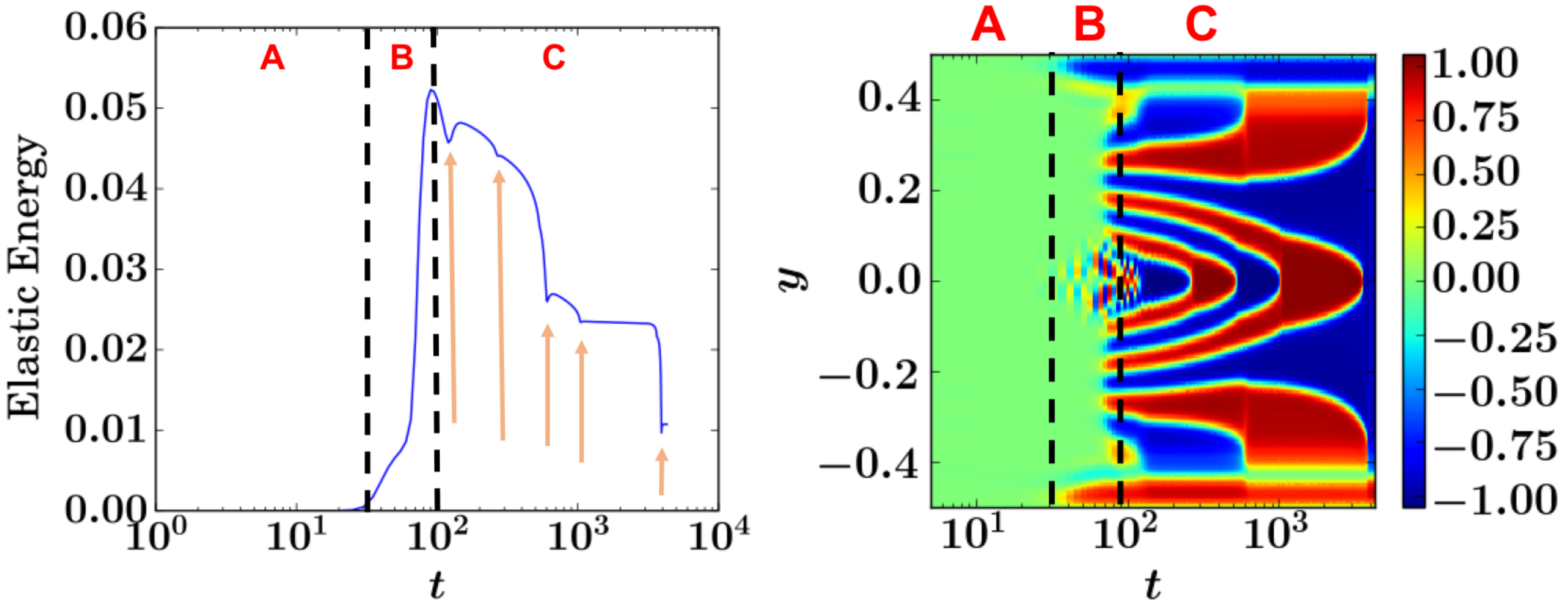
[Fan *et.al.* Phys. Rev. E Rap. Comm. 2017]

- ➢Additional mixing time emerges.

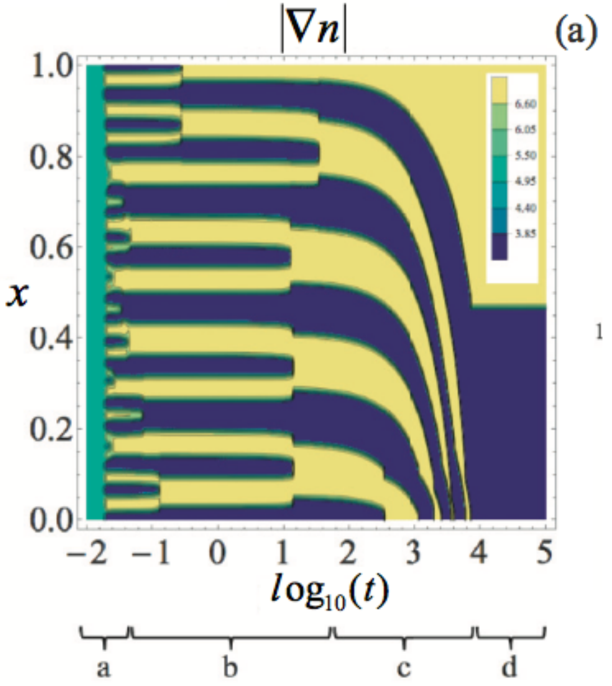
Note coarsening!

Single Eddy Mixing

- ➢The bands merge on a time scale long relative to eddy turnover time.
- ➢The 3 stages are reflected in the elastic energy plot.
- ➢The target bands mergers are related to the dips in the target pattern stage.
- ➢The band merger process is similar to the step merger in drift-ZF staircases.



Episodic relaxation-coarsening Cahn-Hilliard dynamics



[Ashourvan *et.al.* 2016]

Back Reaction – Vortex Disruption

- ➢(MHD only) (A. Gilbert et.al. ‘16; J. Mak et.al. ‘17)
- ➢Demise of kinematic expulsion?
  - Magnetic *tension* grows to react on vorticity evolution!
- ➢Recall: $b \sim B_0(Rm^{1/3})$
  - B.L. field stretched!
- ➢and $\vec{B}\cdot\nabla\vec{B} = -\frac{|B|^2}{r_c}\hat{n} + \frac{d}{ds}(\frac{|B|^2}{2})\hat{t}$
- ➢$\boxed{|\vec{B}\cdot\nabla\vec{B}| \cong b^2/L_0}$

$$\left.\begin{array}{l} r_c \sim L_0 \\ \frac{d}{ds} \sim L_0^{-1} \end{array}\right\} \text{vortex scale}$$

Back Reaction – Vortex Disruption

- So $\rho \frac{d\omega}{dt} = \hat{z} \cdot [\nabla \times (\vec{B} \cdot \nabla \vec{B})]$

  $\rightarrow \rho u \cdot \nabla \omega \sim b^2 / \underset{\uparrow}{l} L_0$

  small BL scale enters

$$v_{A0}^2 = B_0^2 / 4\pi\rho$$

- Feedback → 1 for: $\boxed{Rm \left(\frac{v_{A0}}{u}\right)^2 \sim 1}$

Remember this!

- Critical value to disrupt vortex, end kinematics
- Related Alfven wave emission.
- Note for $Rm \gg 1$ → strong field *not* required
- Will re-appear…

Turbulence

MHD Turbulence – Quick Primer

- (Weak magnetization / 2D)
- Enstrophy conservation broken
- Alfvenic in B$_{rms}$ field – “magneto-elastic” (E. Fermi ‘49)

$$\epsilon = \frac{\langle \tilde{v}^2 \rangle^2}{l^2} \frac{l}{B_{rms}} \quad \Longrightarrow \quad E(k) = (\epsilon B_{rms})^{1/2} k^{-3/2} \text{ (I-K)}$$

- Dual cascade: $\left\{ \begin{array}{l} \text{Forward in energy} \\ \text{Inverse in } \langle A^2 \rangle \sim k^{-7/3} \end{array} \right.$
- What is dominant (A. Pouquet)?
  - conventional wisdom focuses on energy
  - yet $\langle A^2 \rangle$ conservation – freezing-in law!?

Ideal Quadratic Conserved Quantities

- **2D MHD**

1. Energy

$$E = E^K + E^B = \int (\frac{v^2}{2} + \frac{B^2}{2\mu_0}) d^2x$$

2. Mean Square Magnetic Potential

$$H^A = \int A^2 \, d^2x$$

3. Cross Helicity

$$H^C = \int \vec{v} \cdot \vec{B} d^2x$$

- **2D CHNS**

1. Energy

$$E = E^K + E^B = \int (\frac{v^2}{2} + \frac{\xi^2 B_\psi^2}{2}) d^2x$$

2. Mean Square Concentration

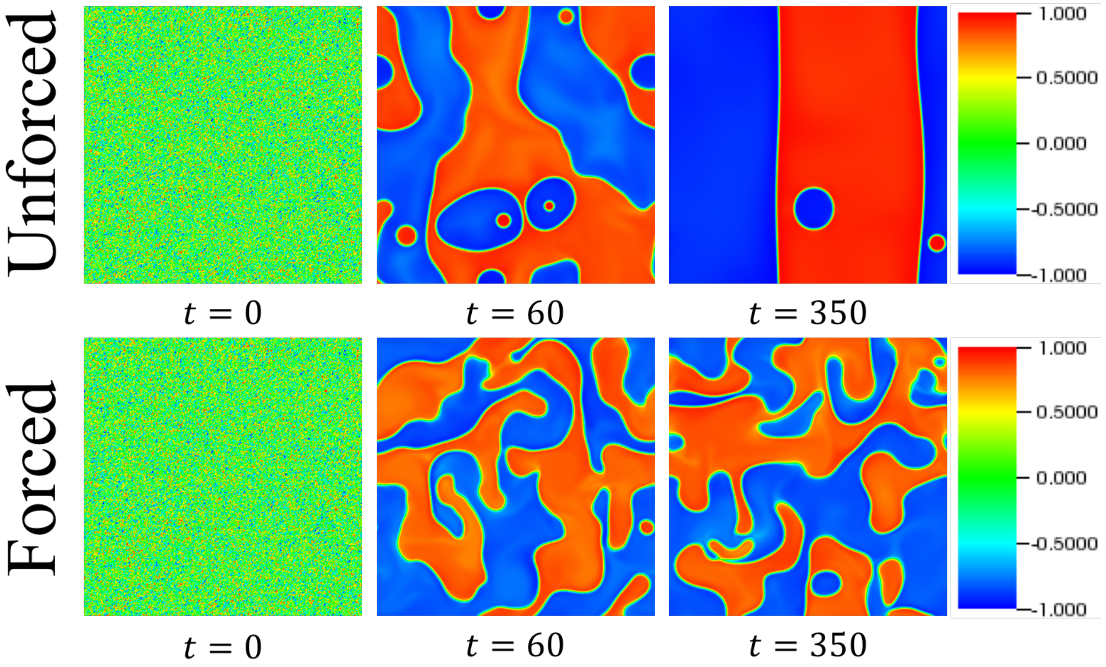
$$H^\psi = \int \psi^2 \, d^2x$$

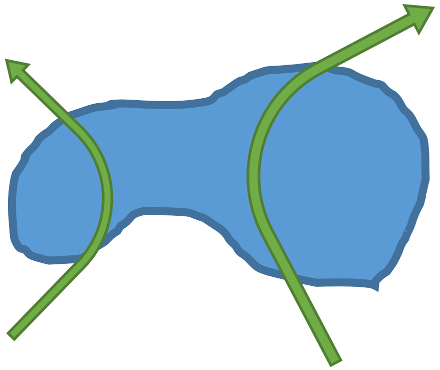
3. Cross Helicity

$$H^C = \int \vec{v} \cdot \vec{B}_\psi \, d^2x$$

Dual cascade expected!

Scales, Ranges, Trends





How big is a raindrop?

- Turbulent straining vs capillarity.
- ρv^2 vs σ / l.

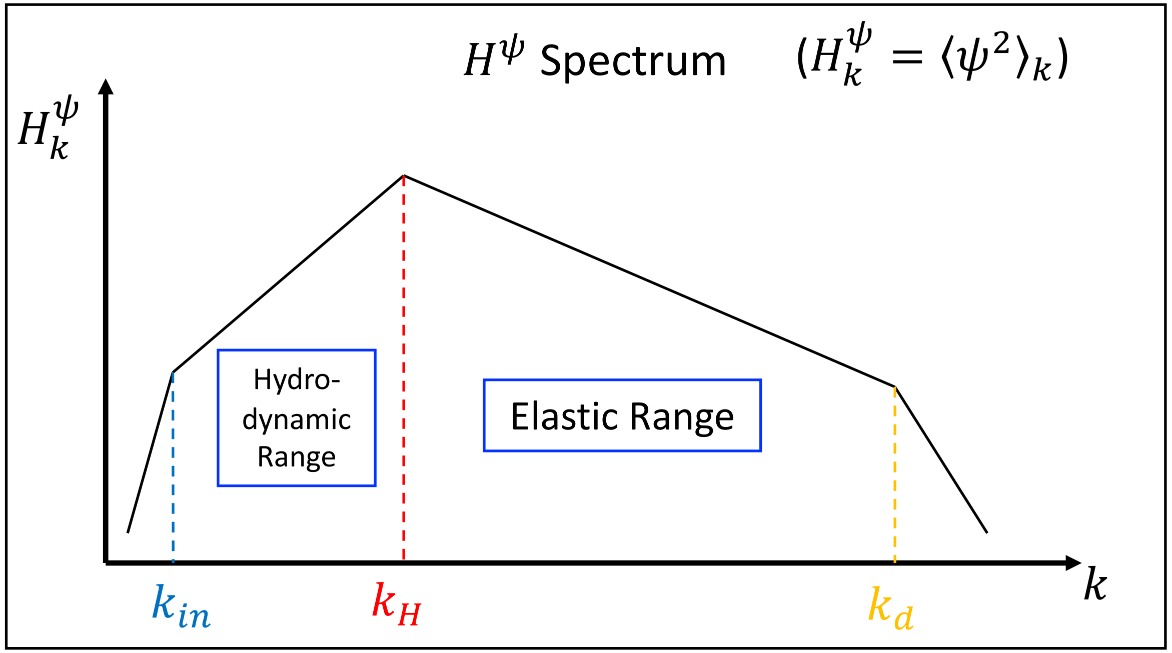
[Hinze 1955]

- ➢Fluid forcing → Fluid straining vs Blob coalescence
- ➢Straining vs coalescence is fundamental struggle of CHNS turbulence
- ➢Scale where turbulent straining ~ elastic restoring force (due surface tension): *Hinze Scale*

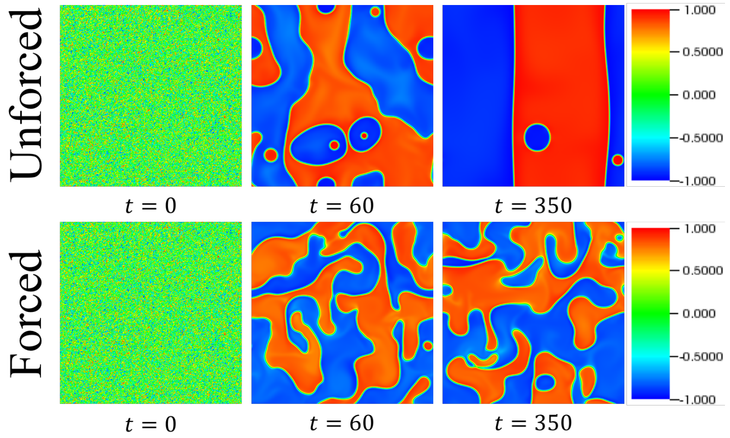
$$L_H \sim \left(\frac{\rho}{\xi}\right)^{-1/3} \epsilon_\Omega^{-2/9}$$

Scales, Ranges, Trends

- ➢Elastic range: $L_H < l < L_d$: where elastic effects matter.
- ➢$L_H/L_d \sim (\frac{\rho}{\xi})^{-1/3}\nu^{-1/2}\epsilon_\Omega^{-1/18}$ → Extent of the elastic range
- ➢$L_H \gg L_d$ required for large elastic range → case of interest

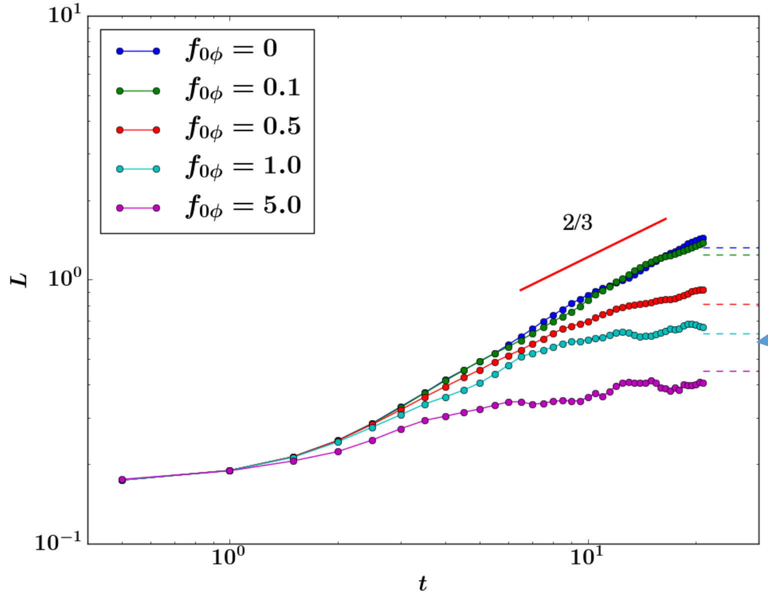


Scales, Ranges, Trends



- Key elastic range physics: ***Blob coalescence***
- Unforced case: $L(t) \sim t^{2/3}$.

  (Derivation: $\vec{v} \cdot \nabla \vec{v} \sim \frac{\xi^2}{\rho} \nabla^2 \psi \nabla \psi \Rightarrow \frac{\dot{L}^2}{L} \sim \frac{\sigma}{\rho} \frac{1}{L^2}$)
- Forced case: blob coalescence arrested at Hinze scale L_H.

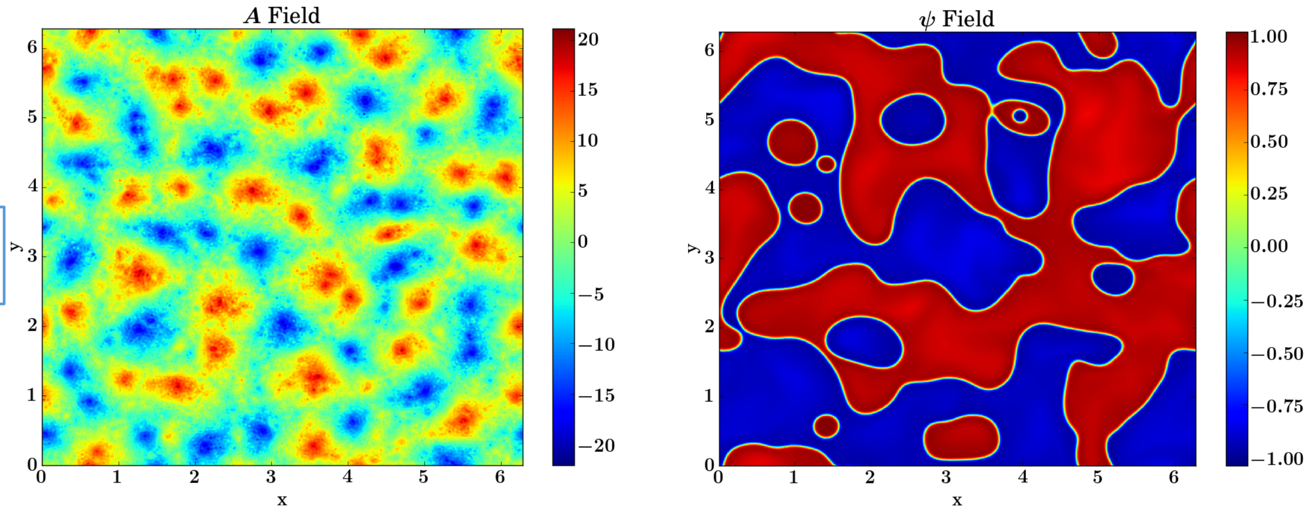


- $L(t) \sim t^{2/3}$ recovered
- Blob growth arrest observed
- Blob growth saturation scale tracks Hinze scale (dashed lines)

- Blob coalescence suggests inverse cascade is fundamental here.

Cascades: Comparing the Systems

MHD



CHNS

- ➢Blob coalescence in the elastic range of CHNS is analogous to flux coalescence in MHD.
- ➢Suggests ***<u>inverse cascade</u>*** of $\langle \psi^2 \rangle$ in CHNS.
- ➢Supported by statistical mechanics studies (absolute equilibrium distributions).
- ➢Arrested by straining.

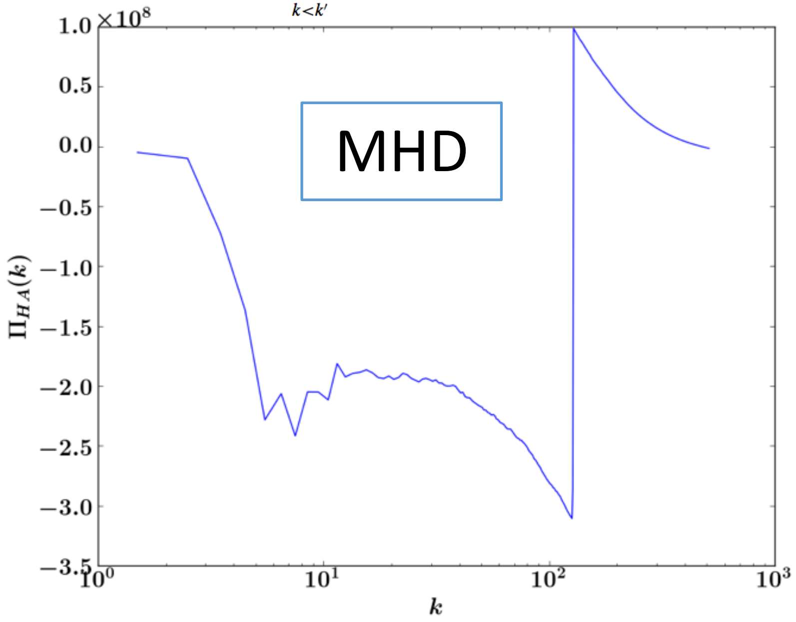
Cascades

- So, *dual cascade*:
  - ***Inverse*** cascade of $\langle\psi^2\rangle$
  - ***Forward*** cascade of E
- Inverse cascade of $\langle\psi^2\rangle$ is formal expression of blob coalescence process → generate larger scale structures till limited by straining
- Forward cascade of E as usual, as elastic force breaks enstrophy conservation
- Forward cascade of energy is analogous to counterpart in 2D MHD

Cascades

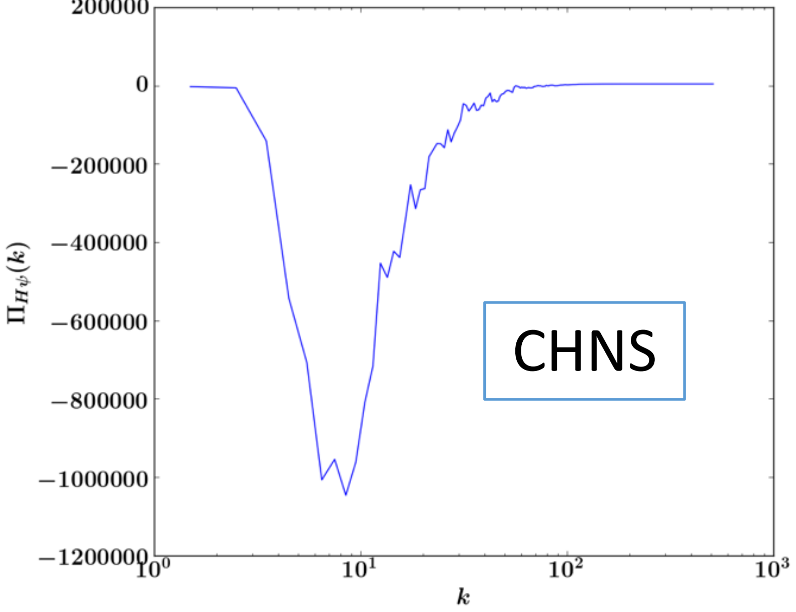
- Spectral flux of $\langle A^2 \rangle$:

$$\Pi_{HA}(k) = \sum_{k<k'} T_{HA}(k'), \quad \text{where } T_{HA}(k) = \langle A_k^*(\mathbf{v}\cdot\nabla A)_k \rangle$$



Spectral flux of $\langle \psi^2 \rangle$:

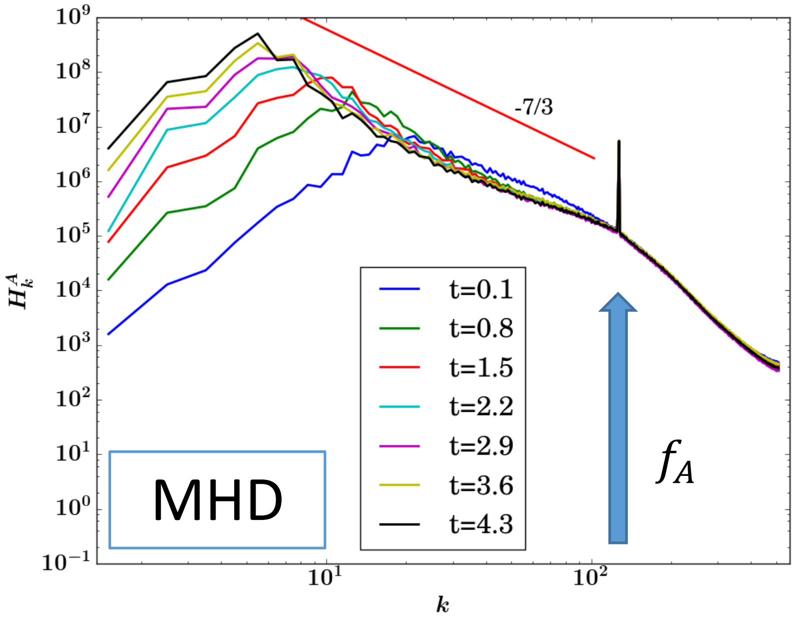
$$\Pi_{H\psi}(k) = \sum_{k<k'} T_{H\psi}(k'), \quad \text{where } T_{H\psi}(k) = \langle \psi_k^*(\mathbf{v}\cdot\nabla \psi)_k \rangle$$



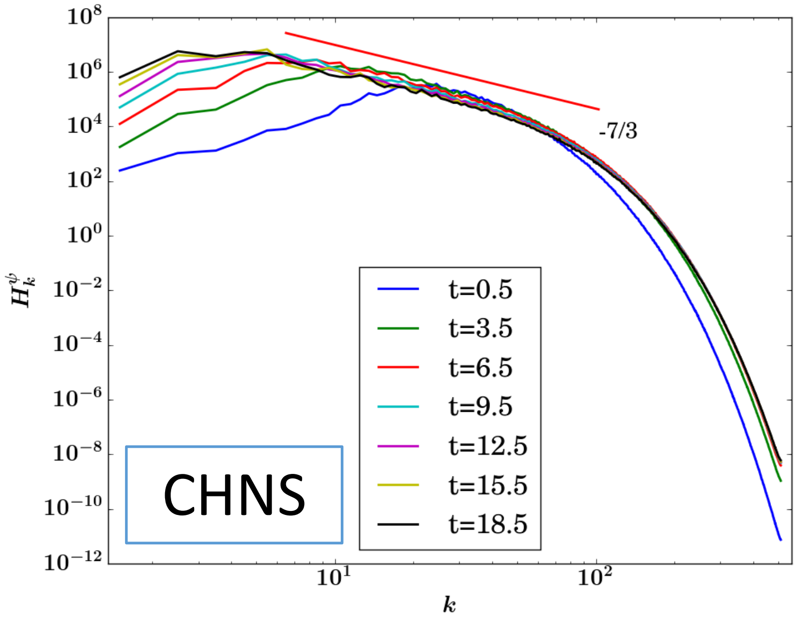
- MHD: weak small scale forcing on A drives inverse cascade
- CHNS: ψ is unforced → aggregates *naturally* ⟺ structure of free energy
- Both fluxes ***negative*** → ***inverse*** cascades

Power Laws

- $\langle A^2 \rangle$ spectrum:



$\langle \psi^2 \rangle$ spectrum:



- Both systems exhibit $k^{-7/3}$ spectra.
- Inverse cascade of $\langle \psi^2 \rangle$ exhibits same power law scaling, so long as $L_H \gg L_d$, maintaining elastic range: Robust process.

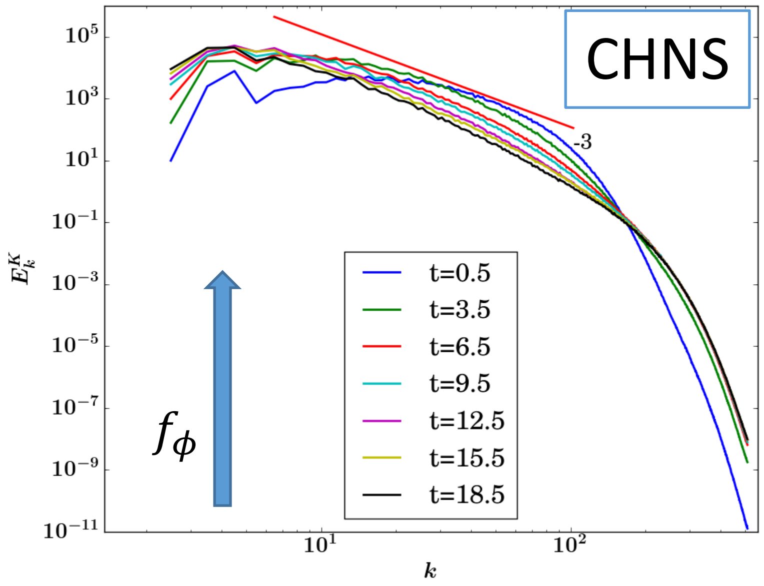
Power Laws

- Derivation of -7/3 power law:
- For MHD, key assumptions:
  - Alfvenic equipartition ($\rho\langle v^2\rangle \sim \frac{1}{\mu_0}\langle B^2\rangle$)
  - Constant mean square magnetic potential dissipation rate ϵ_{HA}, so $\epsilon_{HA} \sim \frac{H^A}{\tau} \sim (H_k^A)^{\frac{3}{2}} k^{\frac{7}{2}}$.
- Similarly, assume the following for CHNS:
  - Elastic equipartition ($\rho\langle v^2\rangle \sim \xi^2\langle B_\psi^2\rangle$)
  - Constant mean square magnetic potential dissipation rate $\epsilon_{H\psi}$, so $\epsilon_{H\psi} \sim \frac{H^\psi}{\tau} \sim (H_k^\psi)^{\frac{3}{2}} k^{\frac{7}{2}}$.

More Power Laws

- Kinetic energy spectrum (Surprise!):
- 2D CHNS: $E_k^K \sim k^{-3}$;
- 2D MHD: $E_k^K \sim k^{-3/2}$.

!



- The -3 power law:
  - Closer to enstrophy cascade range scaling, in 2D Hydro turbulence.
  - Remarkable departure from expected -3/2 for MHD. ***Why?***
- Why does CHNS ←→ MHD correspondence hold well for $\langle\psi^2\rangle_k \sim \langle A^2\rangle_k \sim k^{-7/3}$, yet break down drastically for energy???
- ***What physics*** underpins this surprise??

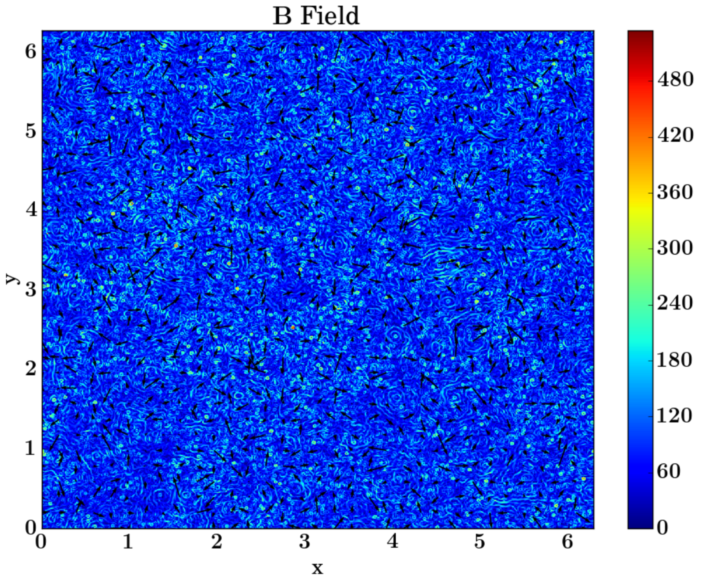
Interface Packing Matters! – Pattern!

- Need to understand ***differences***, as well as similarities, between CHNS and MHD problems.

2D MHD:

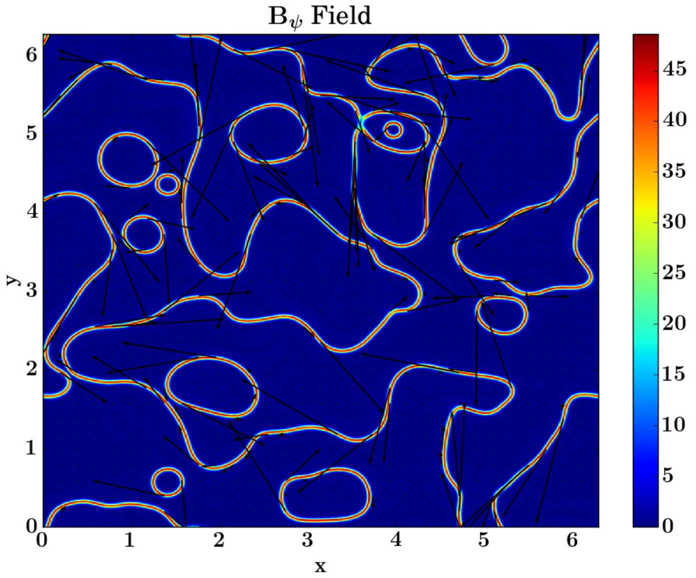
- Fields pervade system.

MHD



2D CHNS:

- Elastic back-reaction is limited to regions of density contrast i.e. $|\vec{B}_\psi| = |\nabla\psi| \neq 0$.
- As blobs coalesce, interfacial region diminished. 'Active region' of elasticity decays.



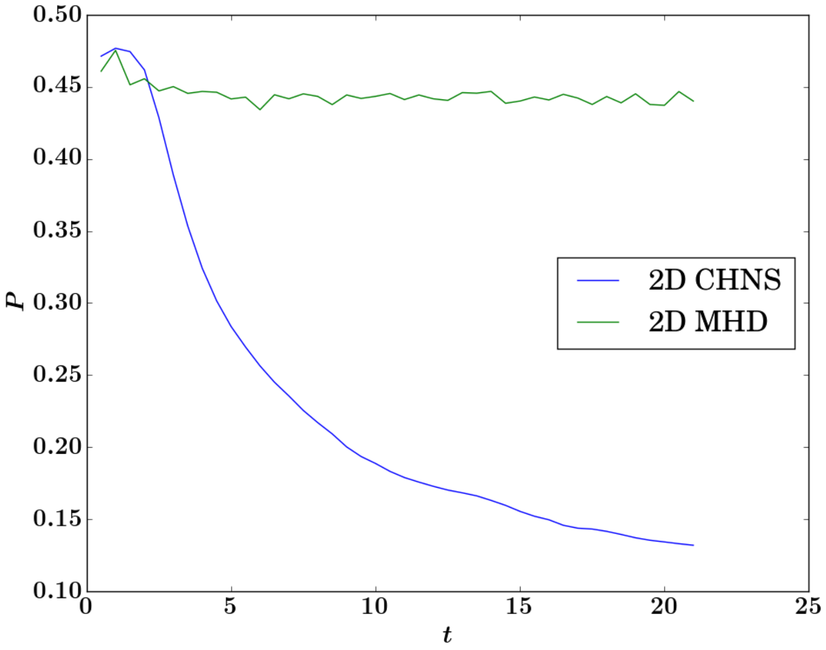
CHNS

Interface Packing Matters!

- Define the ***interface packing fraction*** P:

$$P = \frac{\#\text{ of grid points where } |\vec{B}_\psi| > B_\psi^{rms}}{\#\text{ of total grid points}}$$

  - P for CHNS decays;
  - P for MHD stationary!



- $\partial_t \omega + \vec{v} \cdot \nabla \omega = \frac{\xi^2}{\rho} \vec{B}_\psi \cdot \nabla \nabla^2 \psi + \nu \nabla^2 \omega$: small P → local back reaction is weak.
- Weak back reaction → reduce to 2D hydro → k-spectra
- Blob coalescence coarsens interface network

What Are the Lessons?

- Avoid power law tunnel vision!
- ***Real space*** realization of the flow is necessary to understand key dynamics. Track interfaces and packing fraction P.
- One player in dual cascade (i.e. $\langle\psi^2\rangle$) can modify or constrain the dynamics of the other (i.e. E).
- Against conventional wisdom, $\langle\psi^2\rangle$ inverse cascade due to blob coalescence is the robust nonlinear transfer process in CHNS turbulence.
- Begs more attention to magnetic helicity in 3D MHD.

Transport

Active Scalar Transport

- Magnetic diffusion, ψ transport are cases of active scalar transport
- (Focus: 2D MHD) (Cattaneo, Vainshtein '92, Gruzinov, P. D. '94, '95)

scalar mixing – the usual

$$\partial_t A + \nabla\phi\times\hat{z}\cdot\nabla A = \eta\nabla^2 A$$

$$\partial_t\nabla^2\phi + \nabla\phi\times\hat{z}\cdot\nabla\nabla^2\phi = \nabla A\times\hat{z}\cdot\nabla\nabla^2 A + \nu\nabla^2\nabla^2\phi$$

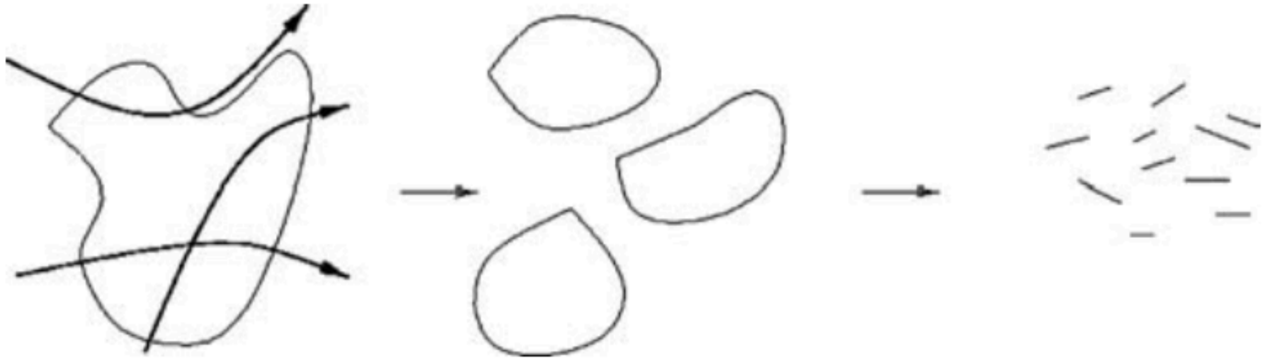
back-reaction

turbulent resistivity

- Seek $\langle v_x A\rangle = -D_T\frac{\partial\langle A\rangle}{\partial x} - \eta\frac{\partial\langle A\rangle}{\partial x}$
- Point: $D_T \neq \sum_{\vec{k}} |v_{\vec{k}}|^2\,\tau_{\vec{k}}^E$, often substantially less
- Why: Memory! ↔ Freezing-in

Origin of Memory?

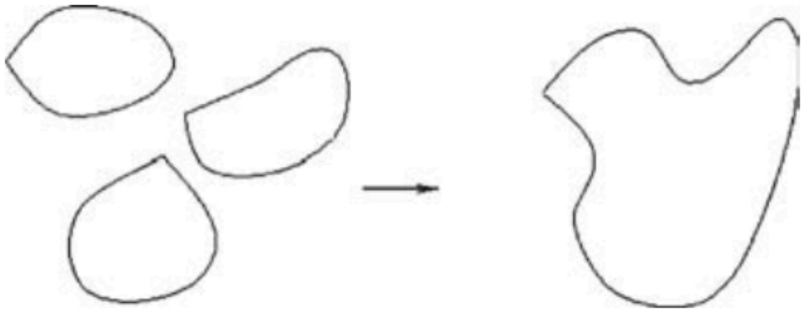
- ➢(a) flux advection vs flux coalescence
  - intrinsic to 2D MHD (and CHNS)
  - rooted in inverse cascade of $\langle A^2 \rangle$
- ➢(b) tendency of (even weak) *mean* magnetic field to "Alfvenize" turbulence [cf: vortex disruption feedback threshold!]
- ➢Re (a): Basic physics of 2D MHD



Forward transfer: fluid eddies chop up scalar A.

Memory Cont'd

- v.s.



Inverse transfer: current filaments and A-blobs attract and coagulate.

- Obvious analogy: straining vs coalescence; CHNS
- Upshot: closure calculation yields:

$$\Gamma_A = -\sum_{\vec{k}'}[\tau_c^{\phi}\langle v^2\rangle_{\vec{k}'} - \tau_c^{A}\langle B^2\rangle_{\vec{k}'}]\frac{\partial\langle A\rangle}{\partial x} + \cdots$$

flux of potential

competition

scalar advection vs. coalescence ("negative resistivity")

(+) (-)

Zeldovich and Alfvenization

- Re (b): Competition winner? → Alfvenization!
- Alfvenization is a natural consequence of stronger $\langle B\rangle$, ala' vortex disruption
- fluid stretches $\langle B\rangle$, ala' $B_0 \rightarrow b$ in flux expulsion
- How to quantify: Zeldovich Theorem

$$H_A = \int d^2x\, H_A = \int d^2x \langle A^2\rangle$$

$$\frac{1}{2}\frac{\partial H_A}{\partial t} = -\Gamma_A \frac{\partial\langle A\rangle}{\partial x} - \eta\langle B^2\rangle$$

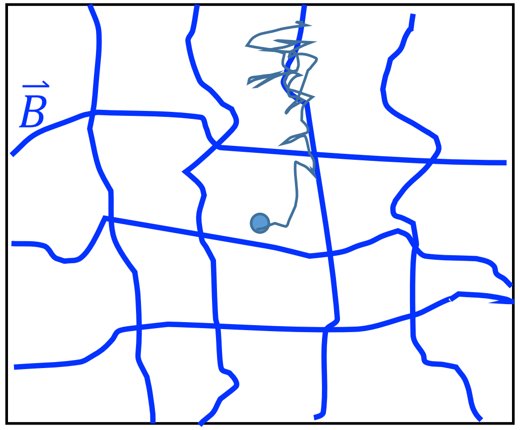
$\uparrow$ production $\uparrow$ dissipation

Zeldovich and Alfvenization, Cont'd

- So $\langle B^2 \rangle \cong -\frac{\Gamma_A}{\eta}\frac{\partial\langle A\rangle}{\partial x} \cong \frac{D_T}{\eta}\left(\frac{\partial\langle A\rangle}{\partial x}\right)^2$ (meta-stationary state)

$$\langle B^2 \rangle \cong \underset{\substack{\uparrow \\ \text{O(Rm)}}}{\frac{D_T}{\eta}} \langle B \rangle^2$$

- Strong RMS field generated from modest $\langle B \rangle$
- Reflects the effect of small scale B-field amplification (i.e. $B_0 \rightarrow b$)
- Ultimately, η asserts itself (Cowling)
- Best think $\langle B^2 \rangle \leftrightarrow T_m$ (elastic energy)



Small scale field as elastic network

Bottom Line

- Eliminate $\langle B^2 \rangle$ in Γ_A using Zeldovich
- So: $\boxed{D_T = D_K / \left[1 + Rm \frac{v_{A0}^2}{\langle v^2 \rangle}\right]}$
- where:
  - D_K is usual kinematic diffusivity
  - $Rm \frac{v_{A0}^2}{\langle v^2 \rangle} \sim 1$ identical to vortex disruption threshold
  - Weak $\langle B \rangle$ "quenches" flux diffusion for large Rm
- Physics is memory enforced by strong, small scale field.

[Implications for α, dynamo, etc.]

(Well-established numerically)

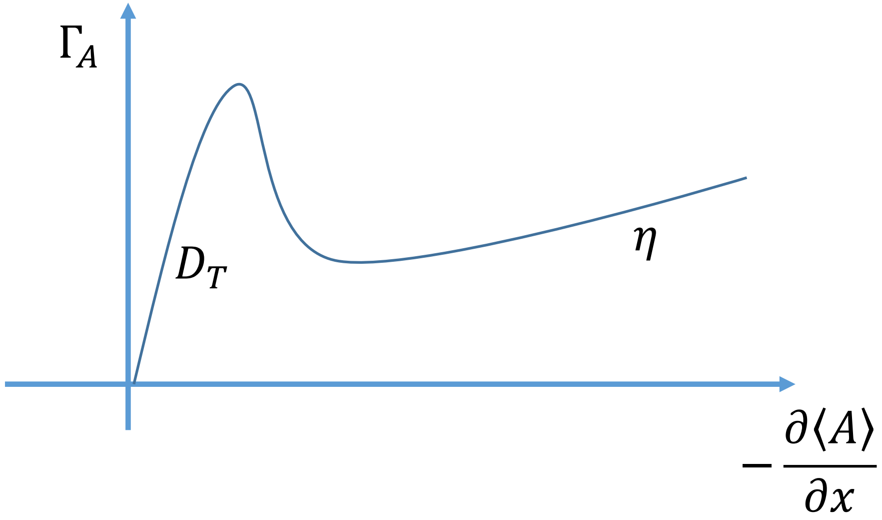
Bottom Line, Cont’d

- ➢ Active scalar transport bifurcation!

$$\Gamma_A = -\frac{D_K \frac{\partial\langle A\rangle}{\partial x}}{\left[1+\frac{Rm}{\rho\langle v^2\rangle}\left(\frac{\partial\langle A\rangle}{\partial x}\right)^2\right]} - \eta\,\frac{\partial\langle A\rangle}{\partial x}$$

(Standard form)

i.e.



Spatio-temporal dynamics largely unexplored

- bi-stable system
- fronts, barriers, domains

- ➢ Expect analogue in CHNS, modulo density gradient

Something Old: Quenching

- ➢$M^2 = \langle \tilde{v}^2 \rangle / v_{A0}^2$
- ➢Higher $v_{A0}^2 / \langle \tilde{v}^2 \rangle \rightarrow$ lower $D_T \rightarrow$ longer E_m persistance
- ➢Ultimately η asserts itself

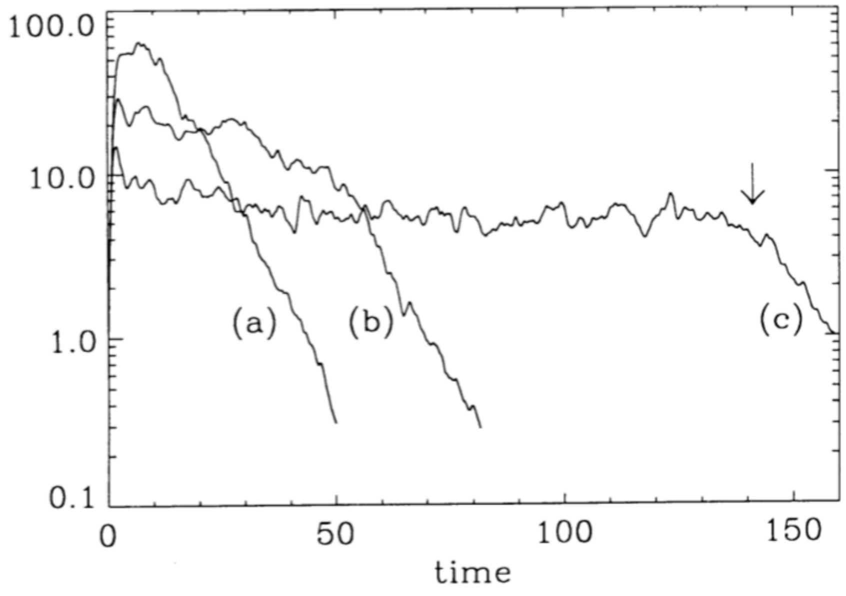
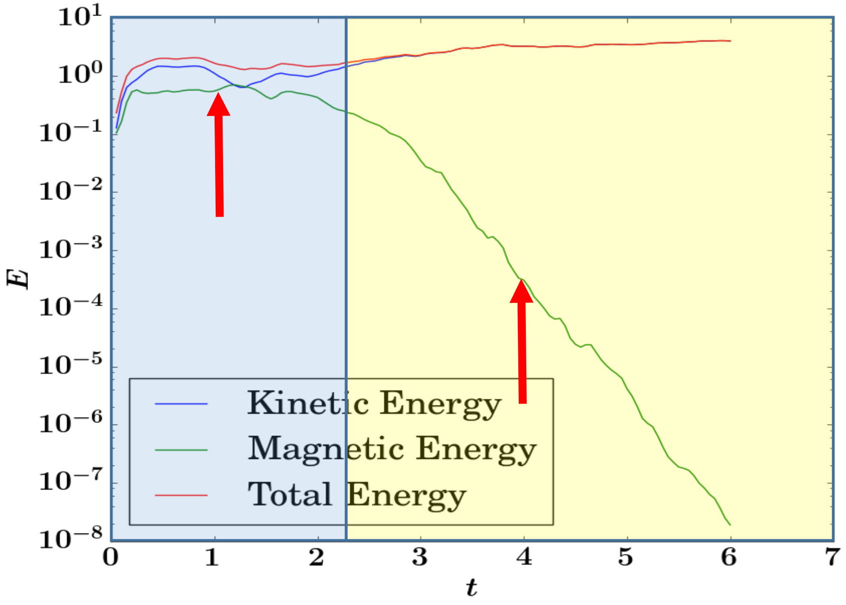


FIG. 3.—Magnetic energy density. Time histories of the total magnetic energy (normalized). The values of M^2 are ∞ for (*a*), 100 for (*b*), and 30 for (*c*).

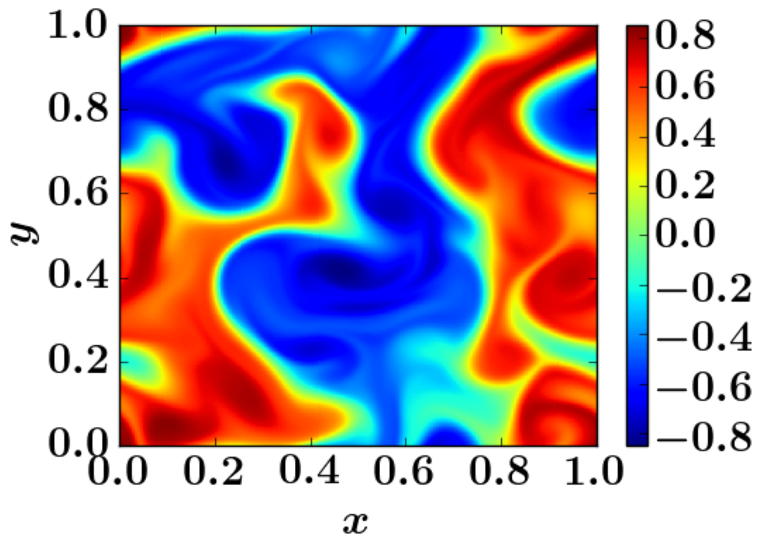
[Cattaneo and Vainshtein '91]

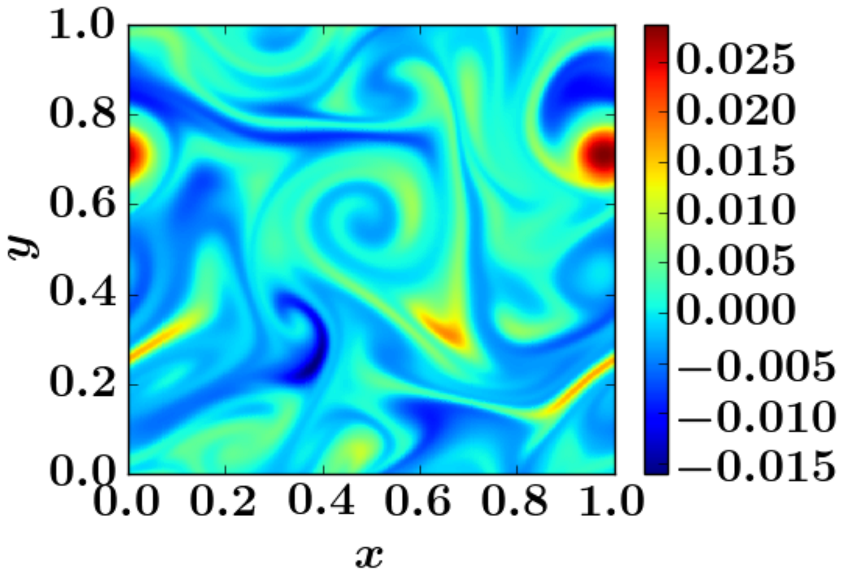
- ➢Blue: $\langle B \rangle$ sufficient for suppression
- ➢Yellow: Ohmic decay phase



Spatial Structure (Preliminary)

- ➢Initial condition: cos(x) for A
- ➢Shorter time (suppression phase)
  - Domains, and domain boundaries evident, resembles CHNS
  - A transport barriers?!
- ➢Longer time (Ohmic decay phase)
  - Well mixed
  - No evidence nontrivial structure

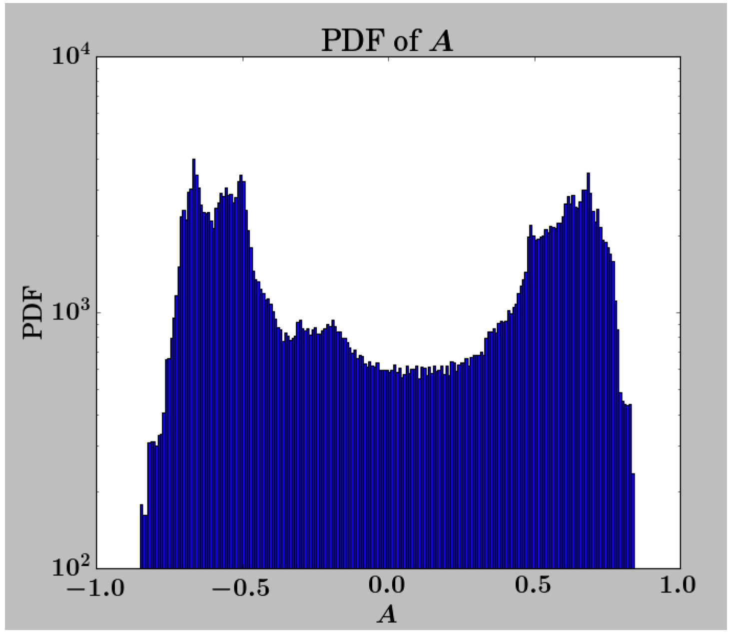


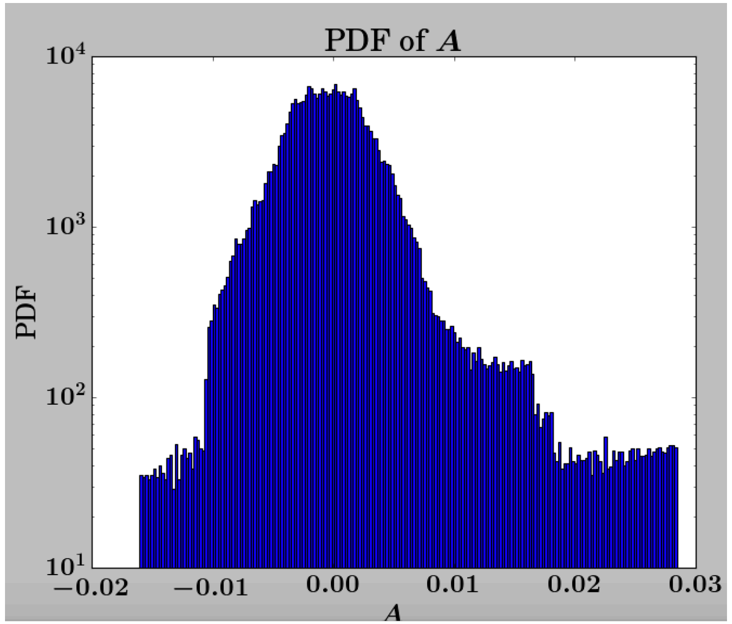


Something New, Cont'd

- For analysis: pdf of A
- Suppression phase:
  - quenched diffusion
  - bi-modal distribution
    - quenching prevents fill-in
    - consequence i.c.

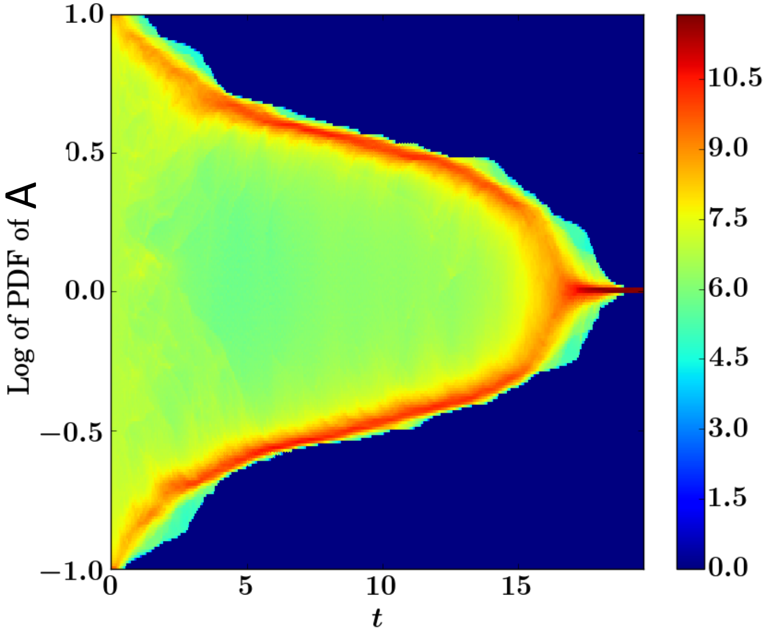
- Ohmic decay phase:
  - uni-modal distribution returns



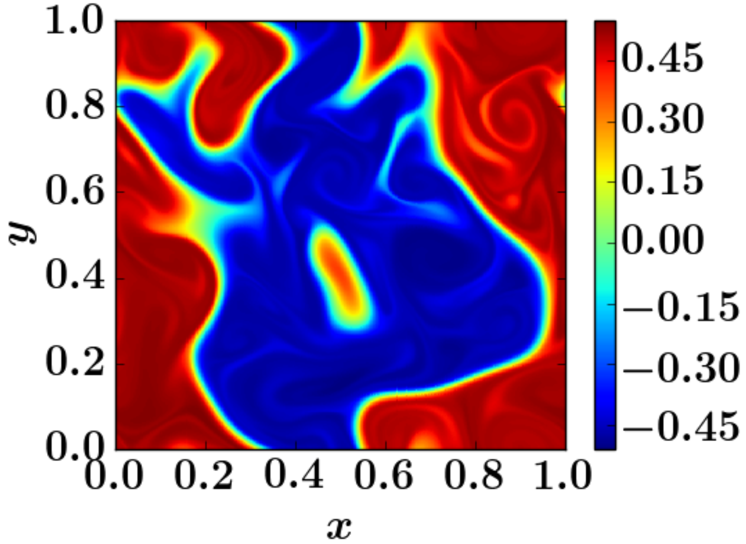


Higher Pm (Lower η_T)

- ➢Bi-modal pdf of A structure persists longer
- ➢Barrier resists Ohmic decay



- ➢A field exhibits strikingly sharp *domain structure*
- ➢Transition layer (barrier) evident
- ➢Clear example of decoupling of transport, intensity.



What of CHNS?

- So far much the same, without Ohmic decay phase
- CH structure feeds elastic energy ↔ resembles forcing in B-field in MHD
- Ongoing

Conclusion

Conclusion, of Sorts

- Elastic fluids ubiquitous, interestingly similar and different. Comparison/contrast is useful approach.
- Simple problems, like flux expulsion (50+ years), reveal a lot about basic feedback dynamics.
- CHNS is interesting example of elastic turbulence where energy cascade is *not* fundamental or dominant.
- Spatio-temporal dynamics of (bi-stable) active scalar transport is a promising direction. Pattern formation in this system is terra novo.
- Revisiting polymer drag reduction would be interesting.