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Turbulence and Transport in Elastic Media: A New Look at Classic Themes

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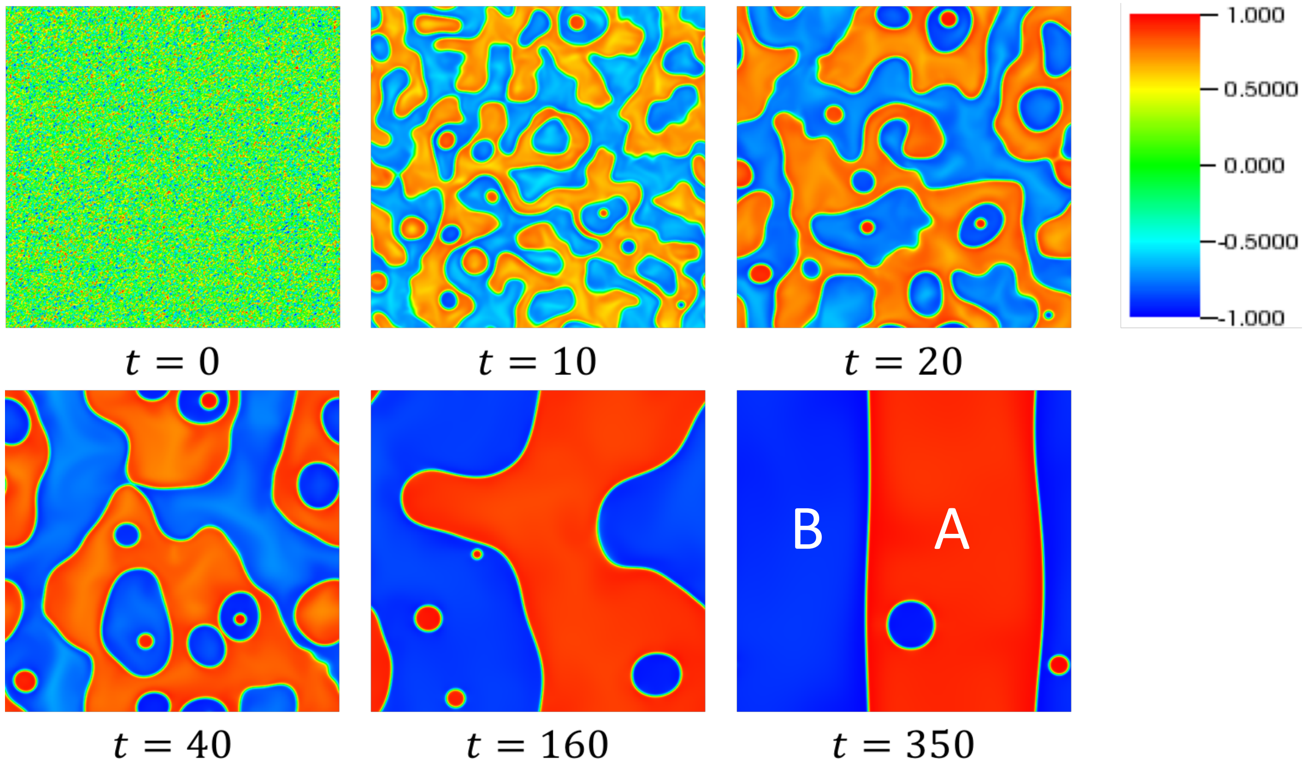
”Tour Guide”

- This is NOT a traditional study of plasma physics.
- It is about a *new* system that is related to systems you are familiar with in plasma physics
- There are many similarities, but some important differences. Watch for these!
- We studied the fundamental physics of cascades and self-organization in this system and in MHD
- It provides a new look at classic themes in plasma physics.

Elastic Media? -- What Is the CHNS System?

- Elastic media – Fluid with internal DoFs → "springiness"
- The Cahn-Hilliard Navier-Stokes (CHNS) system describes ***phase separation*** for binary fluid (i.e. ***Spinodal Decomposition***)



[Fan *et.al*. Phys. Rev. Fluids 2016]

Miscible phase
→ Immiscible phase

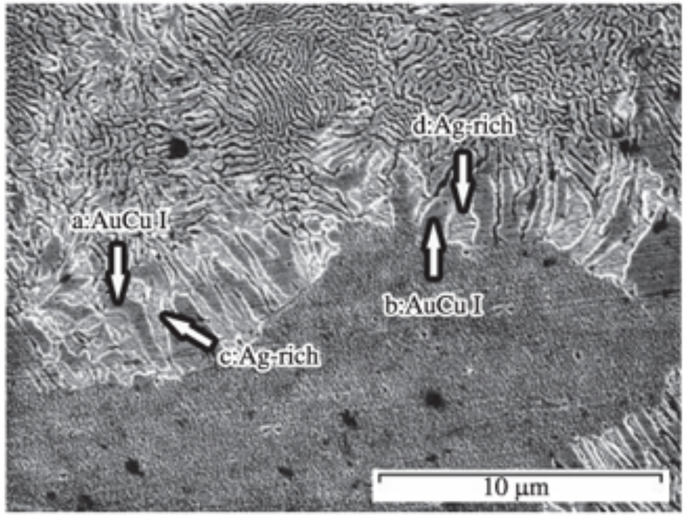


Figure 5. FE-SEM micrograph of specimen aged at 400 °C for 5000 minutes.

[Kim *et.al.* 2012]

Elastic Media? -- What Is the CHNS System?

- How to describe the system: the concentration field
- $\psi(\vec{r},t) \stackrel{\text{def}}{=} [\rho_A(\vec{r},t) - \rho_B(\vec{r},t)]/\rho$: scalar field
- $\psi \in [-1,1]$
- CHNS equations:

$$\partial_t \psi + \vec{v}\cdot\nabla\psi = D\nabla^2(-\psi + \psi^3 - \xi^2\nabla^2\psi)$$

$$\partial_t \omega + \vec{v}\cdot\nabla\omega = \frac{\xi^2}{\rho}\vec{B}_\psi\cdot\nabla\nabla^2\psi + \nu\nabla^2\omega$$

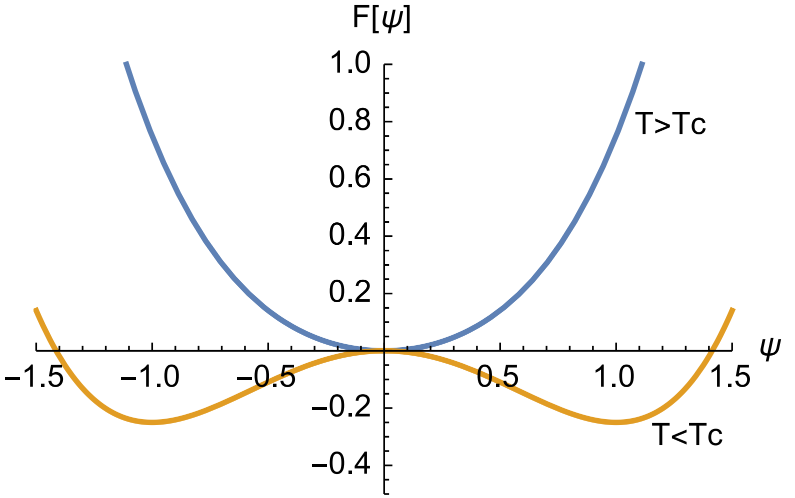
Outline

- A Brief Derivation of the CHNS Model
- 2D CHNS and 2D MHD
- Linear Wave
- Ideal Quadratic Conserved Quantities
- Scales, Ranges, Trends
- Cascades
- Power Laws
- Transport
- Conclusions

A Brief Derivation of the CHNS Model

- Second order phase transition → Landau Theory.
- Order parameter: $\psi(\vec{r},t) \overset{\text{def}}{=} [\rho_A(\vec{r},t) - \rho_B(\vec{r},t)]/\rho$
- Free energy:

$$\mathrm{F}(\psi) = \int d\vec{r}\left(\underbrace{\frac{1}{2}C_1\psi^2 + \frac{1}{4}C_2\psi^4}_{\text{Phase Transition}} + \underbrace{\frac{\xi^2}{2}|\nabla\psi|^2}_{\text{Gradient Penalty}}\right)$$



- $C_1(T), C_2(T)$.
- Isothermal $T < T_C$. Set $C_2 = -C_1 = 1$:

$$\mathrm{F}(\psi) = \int d\vec{r}\left(-\frac{1}{2}\psi^2 + \frac{1}{4}\psi^4 + \frac{\xi^2}{2}|\nabla\psi|^2\right)$$

A Brief Derivation of the CHNS Model

- Continuity equation: $\frac{d\psi}{dt} + \nabla \cdot \vec{J} = 0$. Fick's Law: $\vec{J} = -D\nabla\mu$.
- Chemical potential: $\mu = \frac{\delta F(\psi)}{\delta\psi} = -\psi + \psi^3 - \xi^2\nabla^2\psi$.
- Combining above → Cahn Hilliard equation:

$$\frac{d\psi}{dt} = D\nabla^2\mu = D\nabla^2(-\psi + \psi^3 - \xi^2\nabla^2\psi)$$

- $d_t = \partial_t + \vec{v} \cdot \nabla$. Surface tension: force in Navier-Stokes equation:

$$\partial_t\vec{v} + \vec{v} \cdot \nabla\vec{v} = -\frac{\nabla p}{\rho} - \psi\nabla\mu + \nu\nabla^2\vec{v}$$

- For incompressible fluid, $\nabla \cdot \vec{v} = 0$.

2D CHNS and 2D MHD

- 2D CHNS Equations:

$$\partial_t\psi + \vec{v}\cdot\nabla\psi = D\nabla^2(-\psi + \psi^3 - \xi^2\nabla^2\psi)$$

$$\partial_t\omega + \vec{v}\cdot\nabla\omega = \frac{\xi^2}{\rho}\vec{B}_\psi\cdot\nabla\nabla^2\psi + \nu\nabla^2\omega$$

$-\psi$: Negative diffusion term
ψ^3: Self nonlinear term
$-\xi^2\nabla^2\psi$: Hyper-diffusion term

With $\vec{v}=\hat{z}\times\nabla\phi$, $\omega = \nabla^2\phi$, $\vec{B}_\psi = \hat{z}\times\nabla\psi$, $j_\psi = \xi^2\nabla^2\psi$.

- 2D MHD Equations:

$$\partial_t A + \vec{v}\cdot\nabla A = \eta\nabla^2 A$$

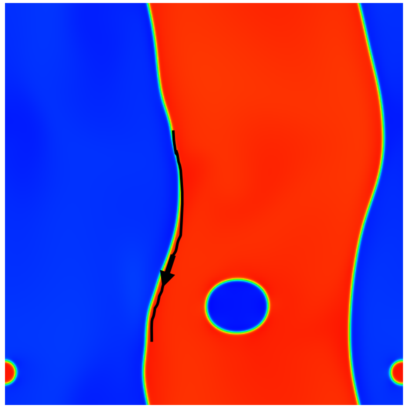
$$\partial_t\omega + \vec{v}\cdot\nabla\omega = \frac{1}{\mu_0\rho}\vec{B}\cdot\nabla\nabla^2 A + \nu\nabla^2\omega$$

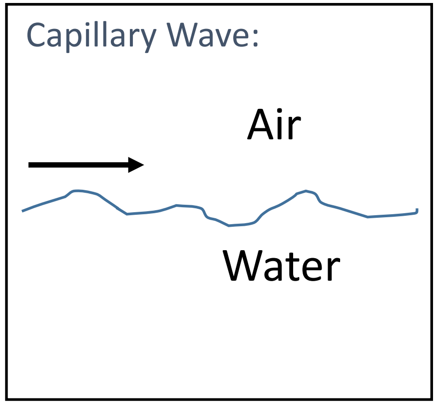
A: Simple diffusion term

With $\vec{v}=\hat{z}\times\nabla\phi$, $\omega = \nabla^2\phi$, $\vec{B} = \hat{z}\times\nabla A$, $j = \frac{1}{\mu_0}\nabla^2 A$.

	2D MHD	2D CHNS
Magnetic Potential	A	ψ
Magnetic Field	$\mathbf{B}$	$\mathbf{B}_\psi$
Current	j	j_ψ
Diffusivity	η	D
Interaction strength	$\frac{1}{\mu_0}$	ξ^2

Linear Wave





- CHNS supports linear "elastic" wave:

$$\omega(k) = \pm\sqrt{\frac{\xi^2}{\rho}}\left|\vec{k}\times\vec{B}_{\psi 0}\right| - \frac{1}{2}i(CD+\nu)k^2$$

Where $C \equiv [-1 - 6\psi_0\nabla^2\psi_0/k^2 - 6(\nabla\psi_0)^2/k^2 - 6\psi_0\nabla\psi_0\cdot i\mathbf{k}/k^2 + 3\psi_0^2 + \xi^2k^2]$

- Akin to capillary wave at phase interface. Propagates ***only*** along the interface of the two fluids, where $|\vec{B}_\psi| = |\nabla\psi| \neq 0$.
- Analogue of Alfven wave.
- Important differences:
  - $\vec{B}_\psi$ in CHNS is large only in the interfacial regions.
  - Elastic wave activity does not fill space.

Ideal Quadratic Conserved Quantities

- **2D MHD**

1. Energy

$$E = E^K + E^B = \int (\frac{v^2}{2} + \frac{B^2}{2\mu_0}) d^2x$$

2. Mean Square Magnetic Potential

$$H^A = \int A^2 \, d^2x$$

3. Cross Helicity

$$H^C = \int \vec{v} \cdot \vec{B} d^2x$$

- **2D CHNS**

1. Energy

$$E = E^K + E^B = \int (\frac{v^2}{2} + \frac{\xi^2 B_\psi^2}{2}) d^2x$$

2. Mean Square Concentration

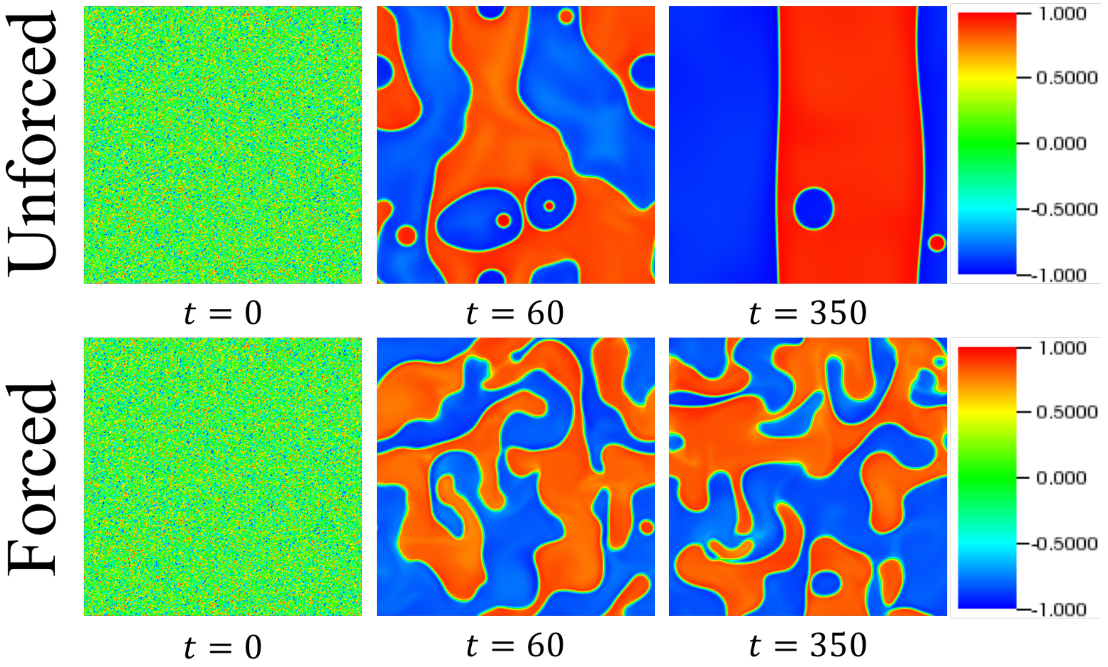
$$H^\psi = \int \psi^2 \, d^2x$$

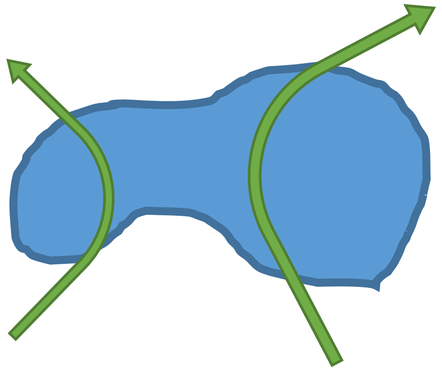
3. Cross Helicity

$$H^C = \int \vec{v} \cdot \vec{B}_\psi \, d^2x$$

Dual cascade expected!

Scales, Ranges, Trends





How big is a raindrop?

- Turbulent straining vs capillarity.
- ρv^2 vs σ / l.

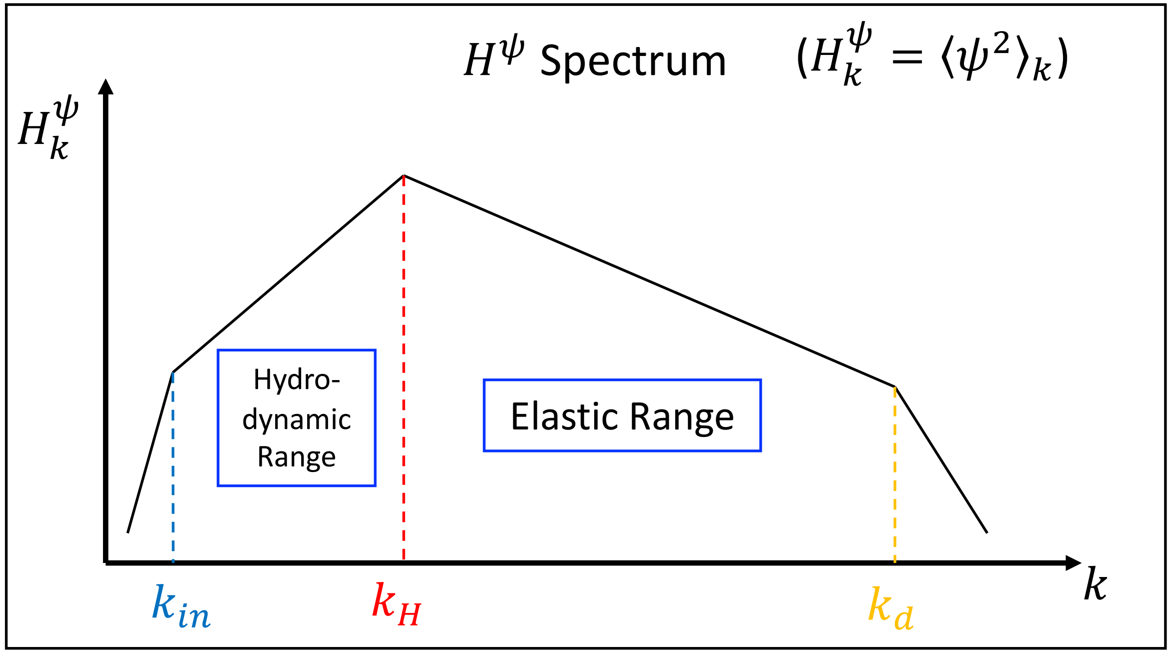
[Hinze 1955]

- Fluid forcing → Fluid straining vs Blob coalescence
- Scale where turbulent straining ~ elastic restoring force (due surface tension): Hinze Scale

$$L_H \sim (\frac{\rho}{\xi})^{-1/3} \epsilon_{\Omega}^{-2/9}$$

Scales, Ranges, Trends

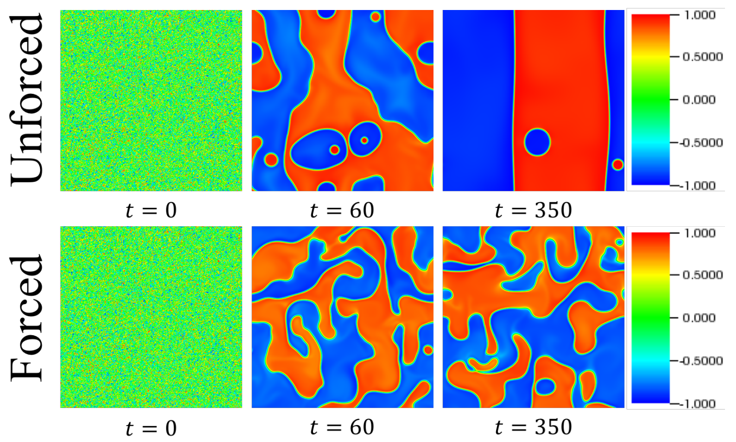
- Elastic range: $L_H < l < L_d$: where elastic effects matter.
- $L_H/L_d \sim (\frac{\rho}{\xi})^{-1/3} \nu^{-1/2} \epsilon_\Omega^{-1/18}$ → Extent of the elastic range
- $L_H \gg L_d$ required for large elastic range → case of interest

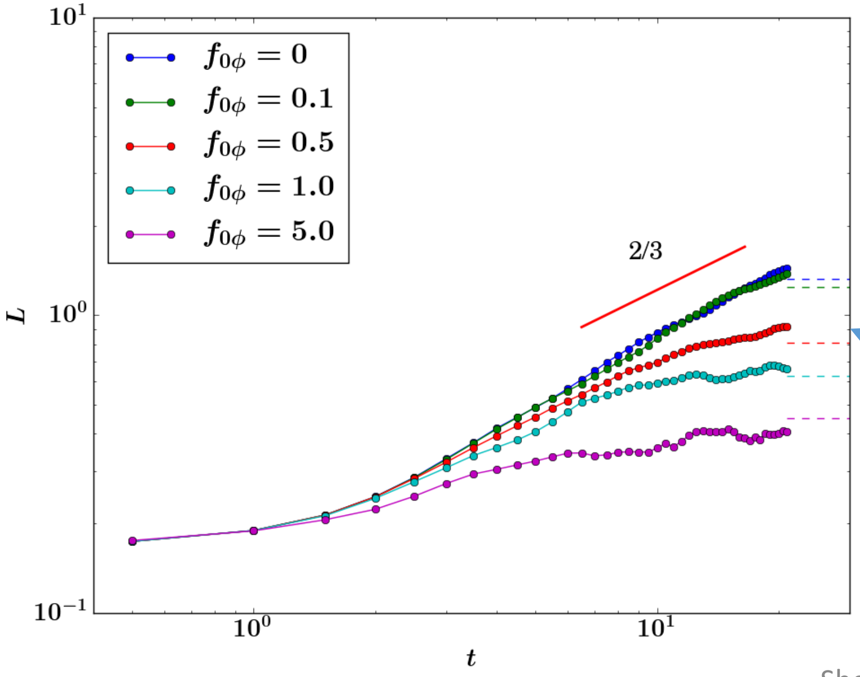


Scales, Ranges, Trends

- Key elastic range physics: ***Blob coalescence***
- Unforced case: $L(t) \sim t^{2/3}$.

  (Derivation: $\vec{v} \cdot \nabla \vec{v} \sim \frac{\xi^2}{\rho} \nabla^2 \psi \nabla \psi \Rightarrow \frac{\dot{L}^2}{L} \sim \frac{\sigma}{\rho} \frac{1}{L^2}$)
- Forced case: blob coalescence arrested at Hinze scale L_H.

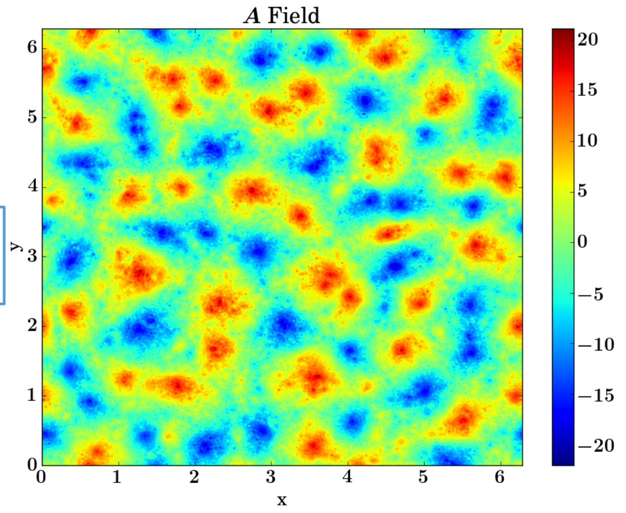


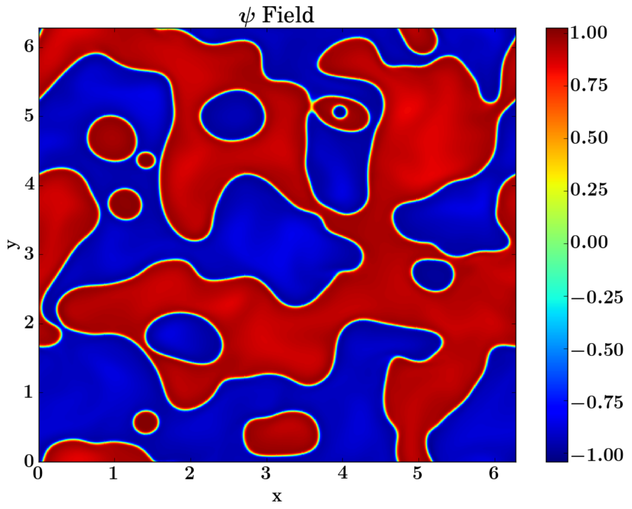


- $L(t) \sim t^{2/3}$ recovered
- Blob growth arrest observed
- Blob growth saturation scale tracks Hinze scale (dashed lines)

Cascades

MHD





CHNS

- Blob coalescence in the elastic range of CHNS is analogous to flux coalescence in MHD.
- Suggests ***inverse cascade*** of $\langle\psi^2\rangle$ in CHNS.
- Supported by the statistical mechanics studies (absolute equilibrium distributions).

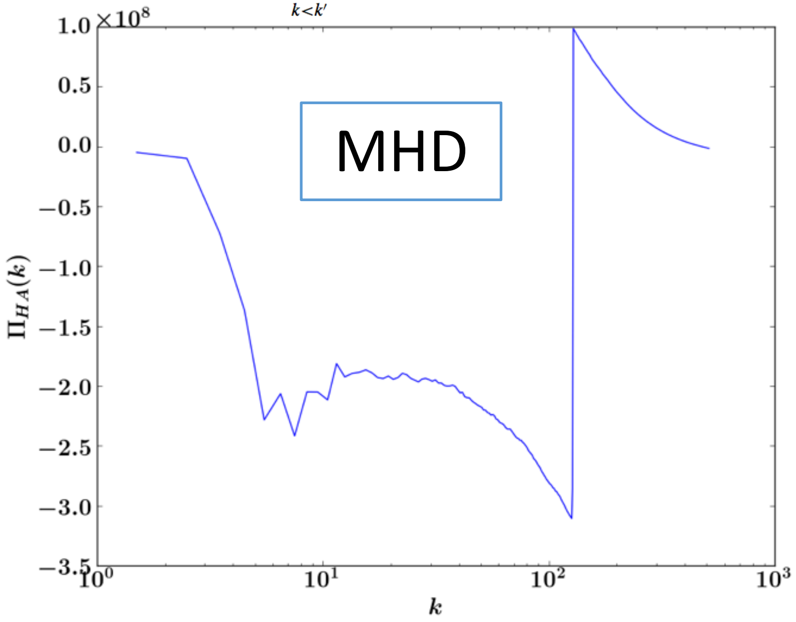
Cascades

- So, dual cascade:
  - ***Inverse*** cascade of $\langle \psi^2 \rangle$
  - ***Forward*** cascade of E
- Inverse cascade of $\langle \psi^2 \rangle$ is formal expression of blob coalescence process → generate larger scale structures till limited by straining
- Forward cascade of E as usual, as elastic force breaks enstrophy conservation

Cascades

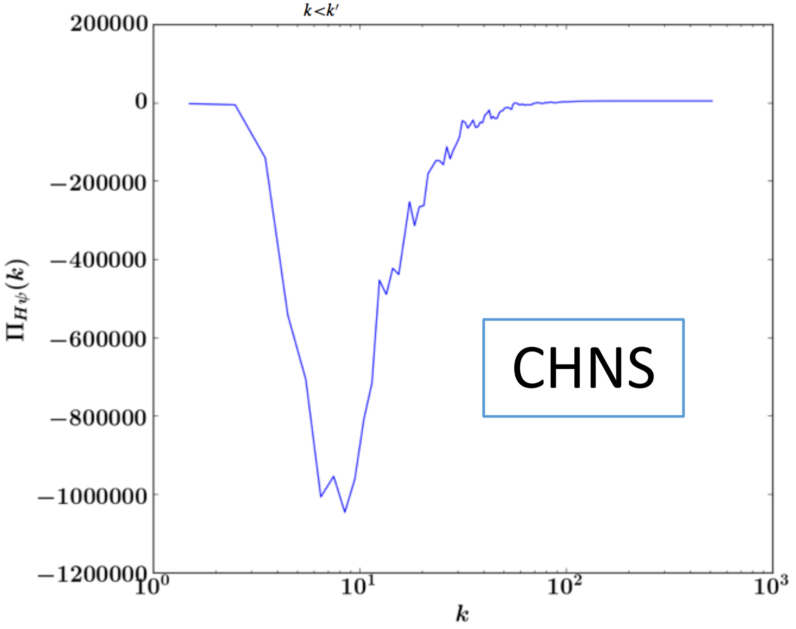
- Spectral flux of $\langle A^2 \rangle$:

$$\Pi_{HA}(k) = \sum_{k<k'} T_{HA}(k'), \quad \text{where } T_{HA}(k) = \langle A_k^* (\mathbf{v} \cdot \nabla A)_k \rangle$$



Spectral flux of $\langle \psi^2 \rangle$:

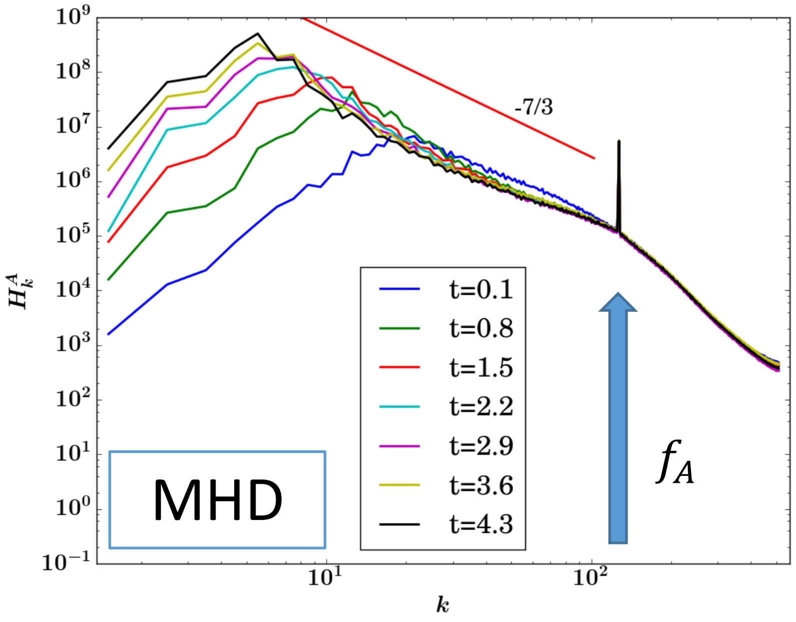
$$\Pi_{H\psi}(k) = \sum_{k<k'} T_{H\psi}(k'), \quad \text{where } T_{H\psi}(k) = \langle \psi_k^* (\mathbf{v} \cdot \nabla \psi)_k \rangle$$

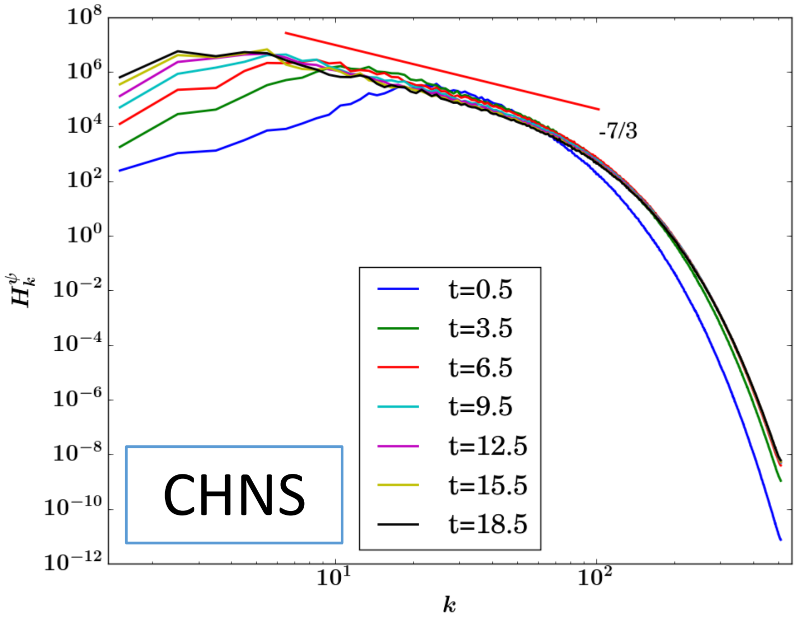


- MHD: weak small scale forcing on A drives inverse cascade
- CHNS: ψ is unforced → aggregates naturally
- Both fluxes ***negative*** → ***inverse*** cascades

Power Laws

- $\langle A^2 \rangle$ spectrum: $\langle \psi^2 \rangle$ spectrum:





- Both systems exhibit $k^{-7/3}$ spectra.
- Inverse cascade of $\langle \psi^2 \rangle$ exhibits same power law scaling, so long as $L_H \gg L_d$, maintaining elastic range: Robust process.

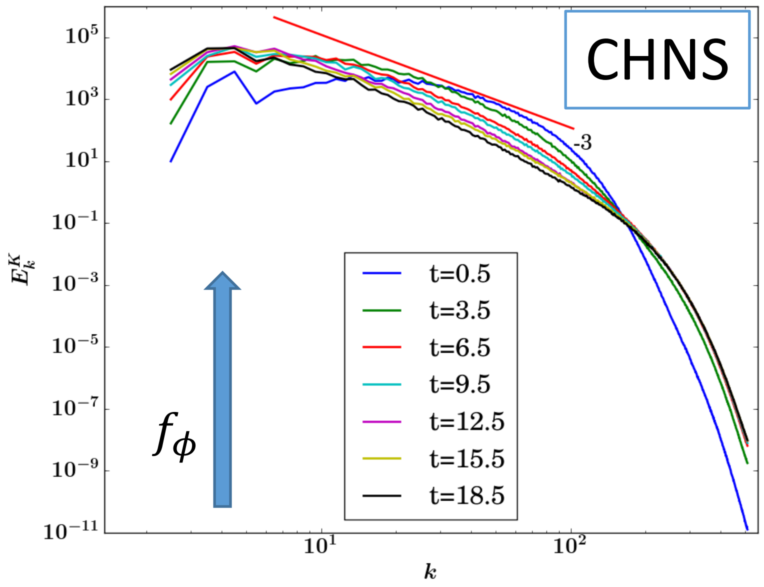
Power Laws

- Derivation of -7/3 power law:
- For MHD, key assumptions:
  - Alfvenic equipartition ($\rho\langle v^2\rangle \sim \frac{1}{\mu_0}\langle B^2\rangle$)
  - Constant mean square magnetic potential dissipation rate ϵ_{HA}, so $\epsilon_{HA} \sim \frac{H^A}{\tau} \sim (H_k^A)^{\frac{3}{2}} k^{\frac{7}{2}}$.
- Similarly, assume the following for CHNS:
  - Elastic equipartition ($\rho\langle v^2\rangle \sim \xi^2\langle B_\psi^2\rangle$)
  - Constant mean square magnetic potential dissipation rate $\epsilon_{H\psi}$, so $\epsilon_{H\psi} \sim \frac{H^\psi}{\tau} \sim (H_k^\psi)^{\frac{3}{2}} k^{\frac{7}{2}}$.

More Power Laws

- Kinetic energy spectrum (Surprise!):
- 2D CHNS: $E_k^K \sim k^{-3}$;
- 2D MHD: $E_k^K \sim k^{-3/2}$.

!



- The -3 power law:
  - Closer to enstrophy cascade range scaling, in 2D Hydro turbulence.
  - Remarkable departure from expected -3/2 for MHD. ***Why?***
- Why does CHNS ←→ MHD correspondence hold well for $\langle\psi^2\rangle_k \sim \langle A^2\rangle_k \sim k^{-7/3}$, yet break down drastically for energy?
- ***What physics*** underpins this surprise?

Interface Packing Matters!

- Need to understand ***differences***, as well as similarities, between CHNS and MHD problems.

2D MHD:

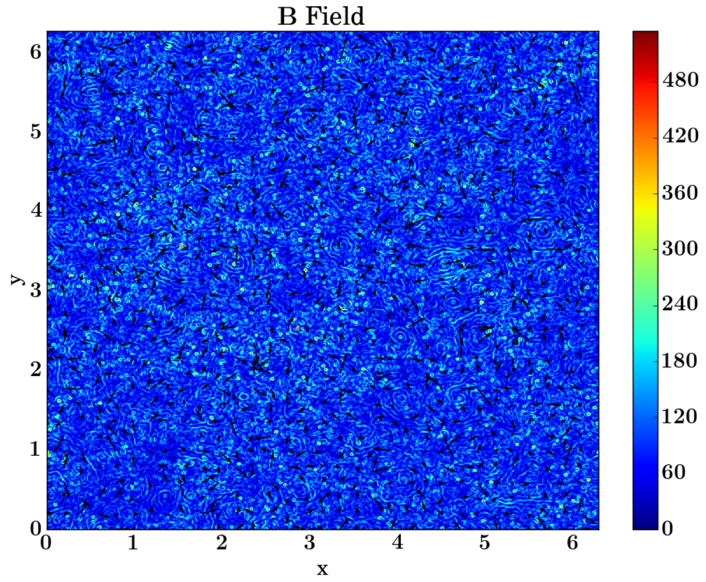
➢ Fields pervade system.

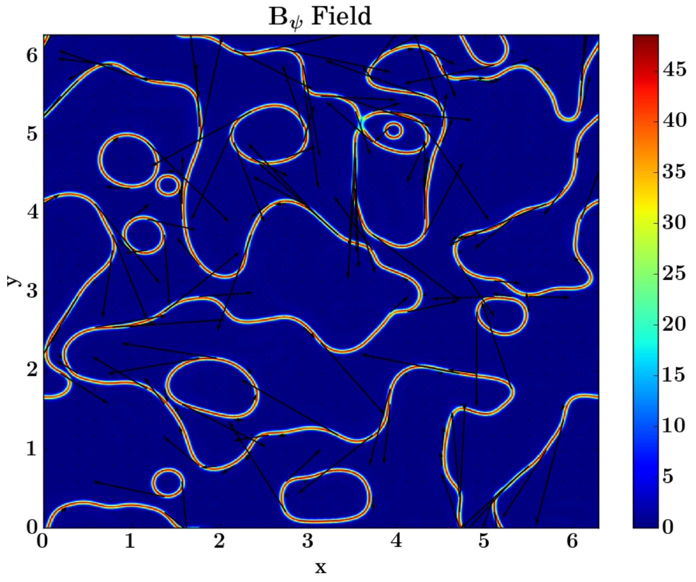
2D CHNS:

➢ Elastic back-reaction is limited to regions of density contrast i.e. $|\vec{B}_\psi| = |\nabla\psi| \neq 0$.

➢ As blobs coalesce, interfacial region diminished. 'Active region' of elasticity decays.

MHD





CHNS

Interface Packing Matters!

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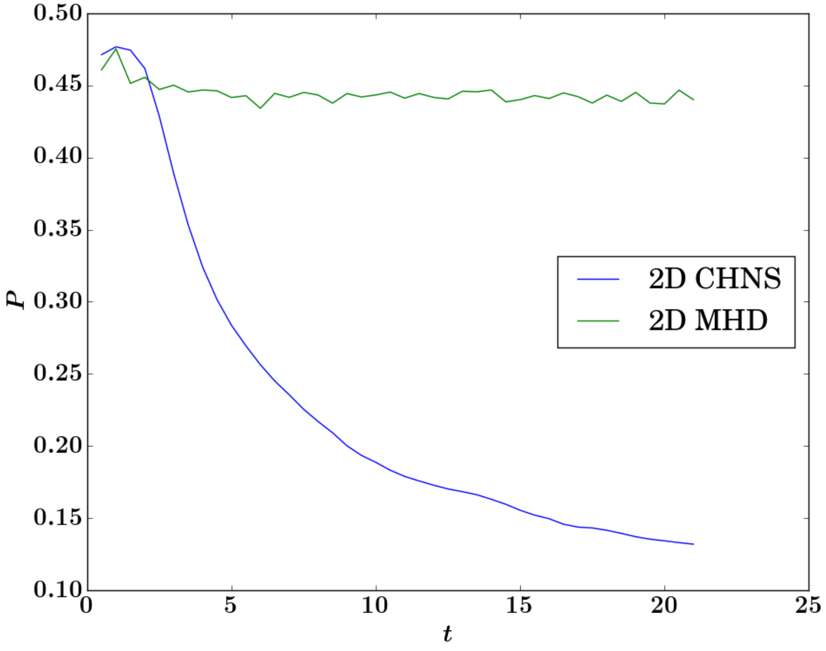
- Define the ***interface packing fraction*** P:

$$P = \frac{\#\ \text{of grid points where } |\vec{B}_\psi| > B_\psi^{rms}}{\#\ \text{of total grid points}}$$

  - ➢ P for CHNS decays;
  - ➢ P for MHD stationary!



- $\partial_t \omega + \vec{v} \cdot \nabla \omega = \frac{\xi^2}{\rho} \vec{B}_\psi \cdot \nabla \nabla^2 \psi + \nu \nabla^2 \omega$: small P → local back reaction is weak.
- Weak back reaction → reduce to 2D hydro

Transport: Something Old

- $M^2 = \langle \tilde{v}^2 \rangle / v_{A0}^2$
- Higher $v_{A0}^2 / \langle \tilde{v}^2 \rangle \rightarrow$ lower $D_T \rightarrow$ longer E_m persistance
- Ultimately η asserts itself

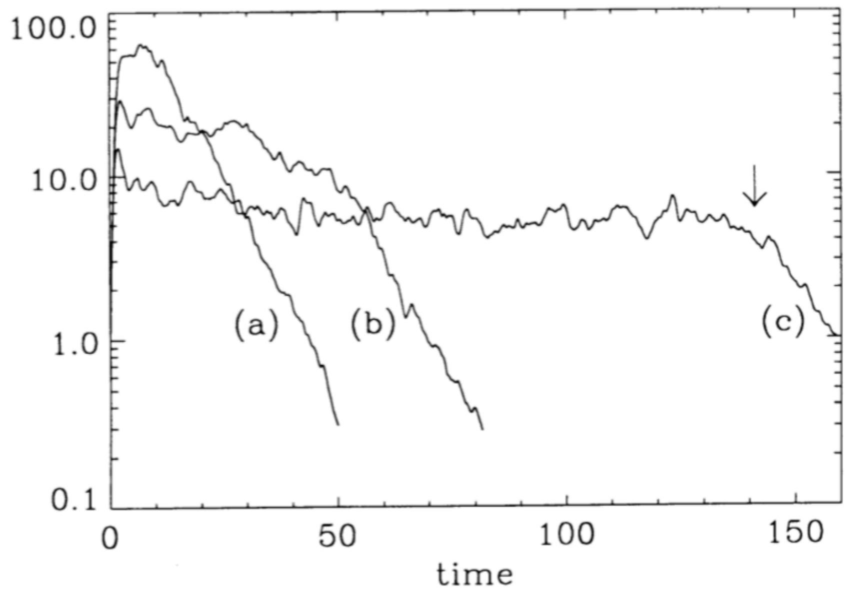
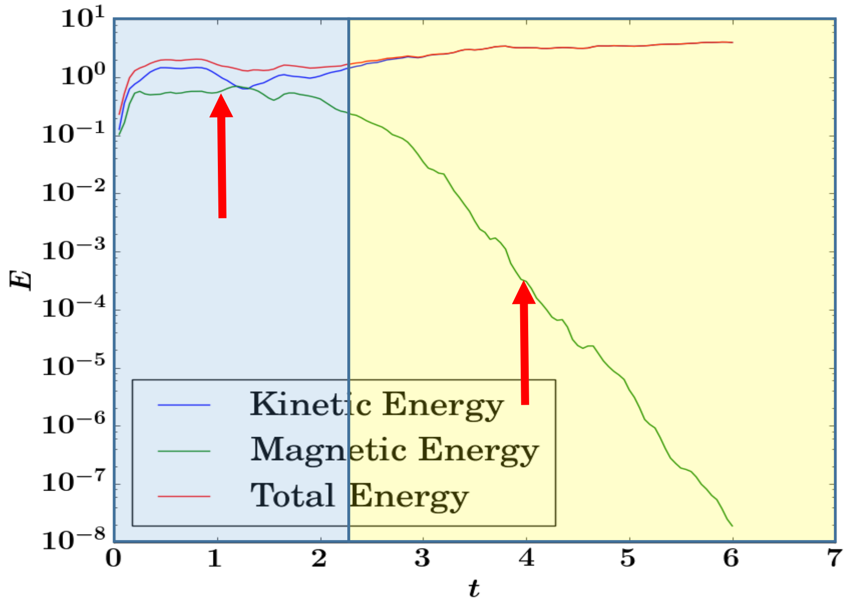


FIG. 3.—Magnetic energy density. Time histories of the total magnetic energy (normalized). The values of M^2 are ∞ for (*a*), 100 for (*b*), and 30 for (*c*).

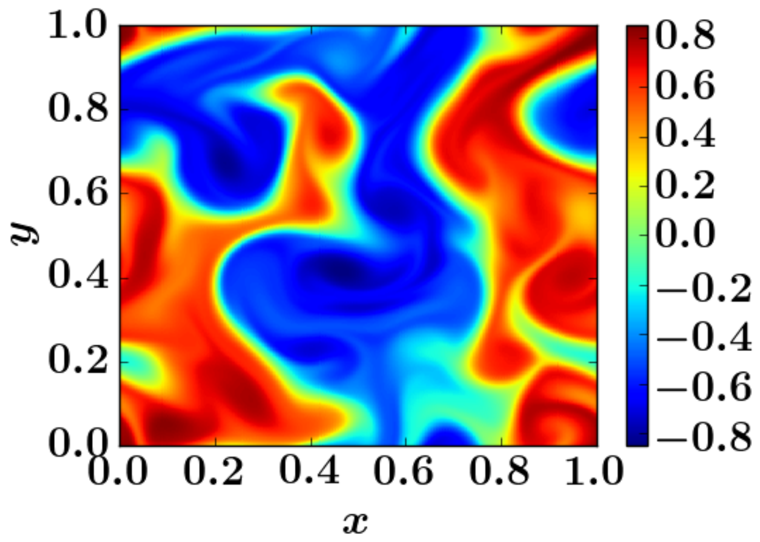
[Cattaneo and Vainshtein '91]

- Blue: $\langle B \rangle$ sufficient for suppression
- Yellow: Ohmic decay phase

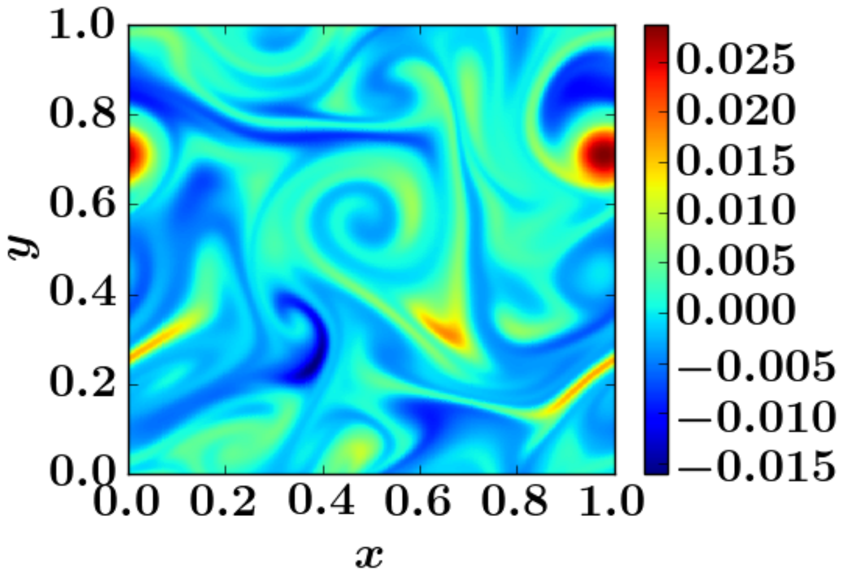


Spatial Structure (Preliminary)

- Initial condition: cos(x) for A
- Shorter time (suppression phase)
  - Domains, and domain boundaries evident, resembles CHNS
  - A transport barriers?!

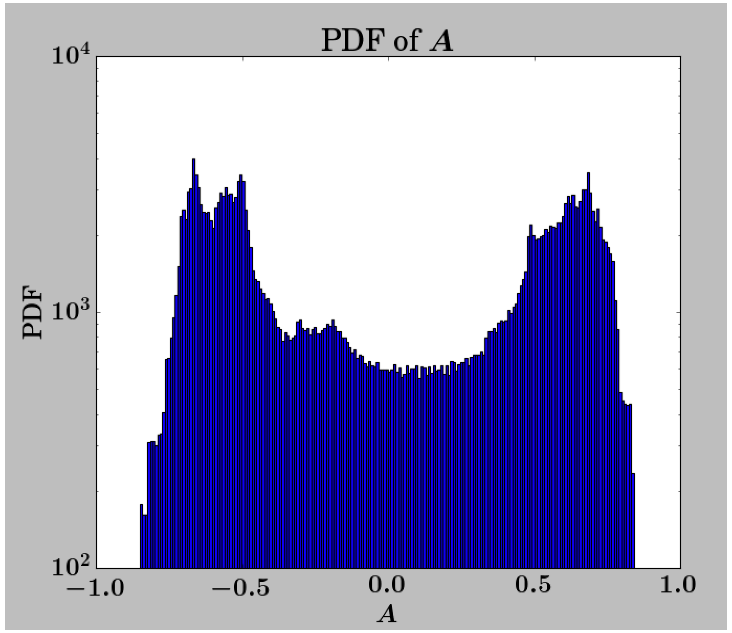


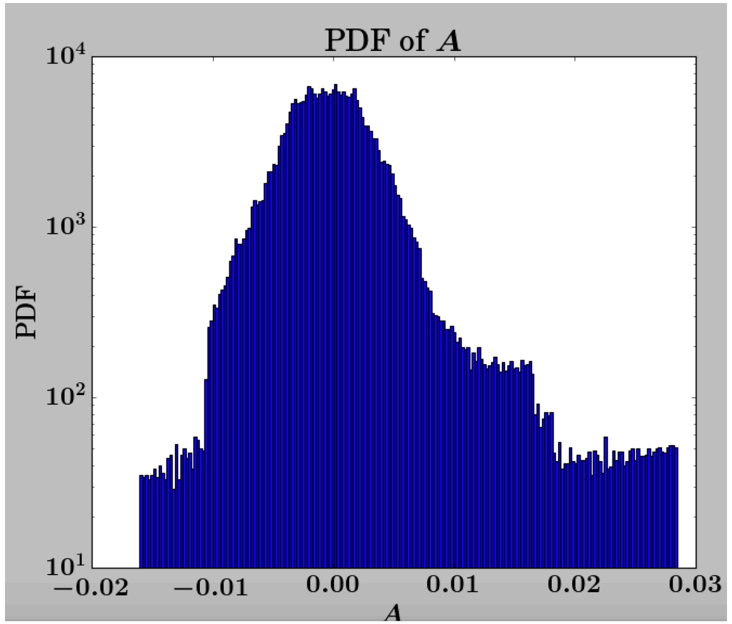
- Longer time (Ohmic decay phase)
  - Well mixed
  - No evidence nontrivial structure



Something New, Cont'd

- For analysis: pdf of A
- Suppression phase:
  - quenched diffusion
  - bi-modal distribution
    - quenching prevents fill-in
    - consequence i.c.
- Ohmic decay phase:
  - uni-modal distribution returns





Conclusion, of Sorts

- Elastic fluids ubiquitous, interestingly similar and different. Comparison/contrast is useful approach.
- CHNS is interesting example of elastic turbulence where energy cascade is *not* fundamental or dominant.
- Spatio-temporal dynamics of (bi-stable) active scalar transport is a promising direction. Pattern formation in this system is terra novo.
- Revisiting polymer drag reduction would be interesting.