Subcritical turbulence spreading and avalanche birth

R.A. Heinonen and P.H. Diamond

CASS and Department of Physics
University of California, San Diego

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Introduction

- Turbulence spreading is an important nonlinear phenomenon in drift wave turbulence
- Challenge the conventional wisdom on spreading and point out issues with the supercritical Fisher equation paradigm
- Suggest a new model based on subcritical turbulence, which features avalanche-like spatiotemporal intermittency
- We make testable predictions which distinguish it from Fisher
- I might say the words ‘phase’ and ‘dynamics’ at some point, but probably not consecutively

Outline

Background: turbulence spreading and avalanching

What the Fick?: turbulence spreading

- Spreading is important because it spells doom for local Fickian transport models
- Turbulence can radially self-propagate (even into linearly stable zones!) via nonlinear coupling

$$\partial_t \varepsilon_{\mathbf{k}} \sim -\sum_{\mathbf{k}'} (\mathbf{k}\cdot\mathbf{k}'\times\hat{z})^2 |\tilde{\phi}_{\mathbf{k}'}|^2 R(\mathbf{k},\mathbf{k}') I_{\mathbf{k}} \rightarrow \frac{\partial}{\partial x} D_x(I_{\mathbf{k}}) \frac{\partial}{\partial x} I_{\mathbf{k}} - \mathbf{kk} : \mathbf{D} I_{\mathbf{k}}$$

$$D_x = \sum_{\mathbf{k}'} k_y'^2 |\phi_{\mathbf{k}'}|^2 R(\mathbf{k},\mathbf{k}')$$

- Decouples flux-gradient relation: local turbulence intensity now depends on global properties of the profiles

Depiction of spreading

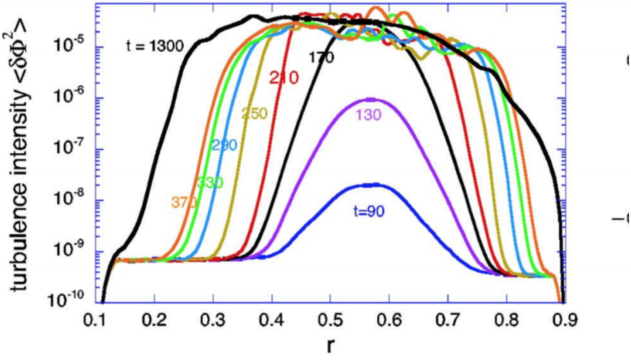


Figure: Spatiotemporal evolution of flux-surface-averaged turbulence intensity in toroidal GK simulation. Linearly unstable region is $0.42 < r < 0.76$. From [Wang et al., 2006]

Avalanches

- Bursty, intermittent transport events associated with SOC. Account for a large percentage of total flux!
- Initially localized fluctuation cascades through neighboring regions via gradient coupling
- Closely related to spreading: both result in fast, mesoscopic turb front propagation. Unified model?

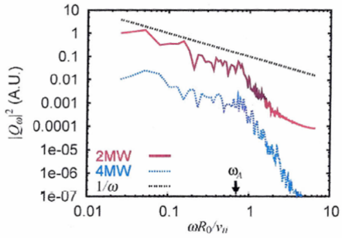


Figure: Heat flux spectrum from GK simulation showing $1/f$ scaling

Depiction of avalanching

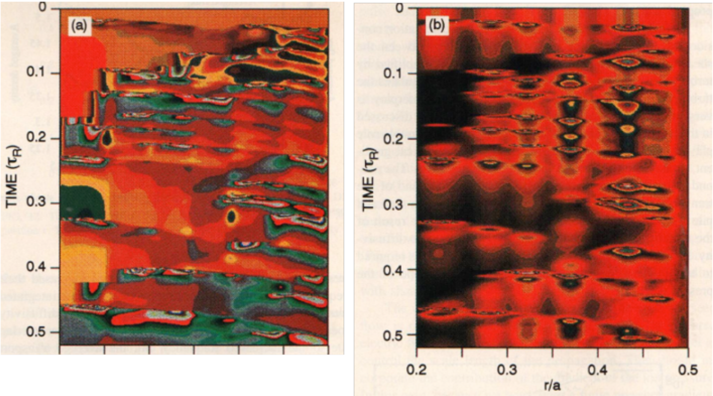


Figure: Pressure (left) and potential (right) contours for simuliations of resistive drift interchange turbulence [Carreras et al., 1996]

Fisher model

- Conventional wisdom for spreading is Fisher-type equation for turbulence intensity:

$$\partial_t I = \underbrace{\gamma_0 I}_{\substack{\text{local lin.}\\ \text{growth/decay}}} - \underbrace{\gamma_{nl} I^2}_{\substack{\text{local nonlin.}\\ \text{coupling to}\\ \text{dissipation}}} + \underbrace{\partial_x(D_0 I \partial_x I)}_{\text{nonlin. diffusion of turb. energy}}$$

- For $\gamma_0 > 0$, dynamics characterized by traveling fronts connecting unstable "laminar root" $I = 0$ and saturated "turbulent root" $I = \gamma_0/\gamma_{nl}$ with speed $c = \sqrt{\frac{D_0\gamma_0^2}{2\gamma_{nl}}}$

Depiction of Fisher evolution

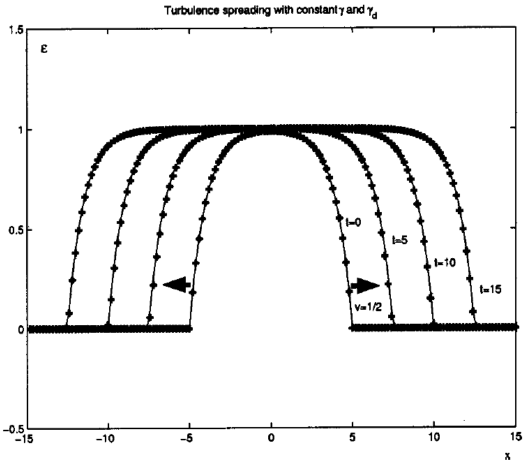


Figure: Evolution of traveling turbulence front in Fisher model. From [Gürcan and Diamond, 2006]

How does Fisher do?

- Propagation speed and characteristic front size $\ell \sim \sqrt{D/\gamma_0}$ in reasonable agreement with simulation
- Can be derived with some rigor from Fokker-Planck approach or renormalization of Hasegawa-Wakatani [Gürcan and Diamond, 2005, Gürcan and Diamond, 2006]
- But: weak spreading into stable zone. Dubiously consistent with experiment?

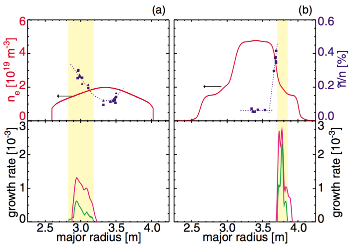


Figure: Experiment by Nazikian et al 2005 clearly showing fluctuations in stable zone

When does Fisher even make sense?

- Fisher model purports to describe spreading of a patch of turbulence in linearly unstable zone
- Begs the question: *why didn't noise already excite the whole system to turbulence?*
- Only relevant if $\gamma_0 \ll c/\Delta x$ i.e. $\Delta x^2 \gamma_{nl} \ll D_0$
- Otherwise, physical fronts separating laminar/turbulent domains generally require *bistability* à la [Pomeau, 1986]

Bistable model

A new(ish) model is born

- Heinonen and Diamond 2019: propose phenomenological model of form

$$\partial_t I = \gamma_1 I + \gamma_2 I^2 - \gamma_3 I^3 + \partial_x (D(I) \partial_x I)$$

- take $D(I) = D_0 I$
- New physics: nonlinear turbulence drive $\propto I^2$. Can sustain sufficiently large fluctuations even when linearly damped
- *Bistable* in weak damping regime
- Estimate $\gamma_1 \sim \epsilon \omega_*,\ \gamma_{2,3} \sim \omega_*,\ D_0 \sim \chi_{GB}$
- But is MF plasma actually subcritically unstable?

Evidence for subcriticality

- [Inagaki et al., 2013]: experiments demonstrate hysteresis between fluctuation intensity and driving gradient (no TB present). Suggests bistable S-curve relation?
- Turbulence subcritical in presence of strong perpendicular flow shear [Carreras et al., 1992, Barnes et al., 2011, van Wyk et al., 2016] or in the presence of magnetic shear [Biskamp and Walter, 1985, Drake et al., 1995]
- Profile corrugations [Waltz, 1985, Waltz, 2010] and phase space structures [Lesur and Diamond, 2013] can drive nonlinear instability

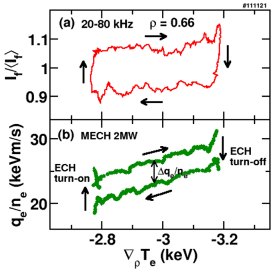


Figure: Hysteresis between intensity and gradient, flux and gradient

Cousin models

- Compare to bistable models for subcritical transition to fluid turbulence [Barkley et al., 2015, Pomeau, 2015].
- Compare to [Gil and Sornette, 1996] model for sandpile avalanches

$$\partial_t S = \gamma\left(|\partial_x h|/g_c - 1\right) S + \beta S^2 - S^3 + \partial_x(D_S S \partial_x S)$$
$$\partial_t h = \partial_x(D_h S \partial_x h).$$

- $S \leftrightarrow I$, $h \leftrightarrow p$
- Weak gradient coupling limit $D_p \ll D_I \Rightarrow$ our model
- Strong gradient coupling limit: I slaved to p. $\partial_x p \propto I^{-1} \Rightarrow$ linear term is $c - \gamma I$, where c is a constant which depends on BCs. Bistable again!

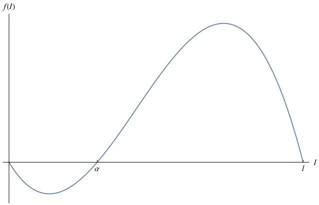
Model analysis I

$$\partial_t I = \gamma_1 I + \gamma_2 I^2 - \gamma_3 I^3 + \partial_x (D(I)\partial_x I)$$

- Qualitatively similar to Fisher EXCEPT in weak damping case $\gamma_1 < 0$ and $\gamma_2^2 > 4|\gamma_1|\gamma_3$
- Can then transform to Zel'dovich/Nagumo equation

$$\partial_t I = f(I) + \partial_x (DI\partial_x I)$$
$$f(I) \equiv \gamma I(I - \alpha)(1 - I)$$

where $\alpha \equiv I_-/I_+,\ \gamma \equiv I_+^2\gamma_3,\ D \equiv I_+ D_0,\ I_\pm \equiv (\gamma_2 \pm \sqrt{\gamma_2^2 - 4|\gamma_1|\gamma_3})/2\gamma_3$



Model analysis II

- Can write in variational form

$$D(I)\partial_t I = -\frac{\delta \mathcal{F}}{\delta I}$$

with free energy functional

$$\mathcal{F} = \int dx\, [\underbrace{\frac{1}{2}(D(I)\partial_x I)^2}_{\text{kinetic/flux}} \underbrace{- \int_0^I dI'\, D(I') f(I')]}_{\text{potential}}$$

and $d\mathcal{F}/dt \leq 0$

Model analysis III

- $I = 0$ metastable for $\alpha < \alpha^* = 3/5$, abs. stable for $\alpha > \alpha^*$
- "Potential barrier" at $I = \alpha$: threshold for onset of nonlinear instability

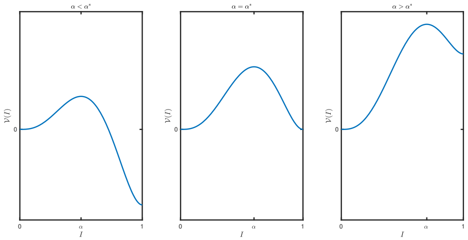


Figure: "Potential" part of $\mathcal{F}$

Model analysis IV

- Unlike Fisher, traveling fronts admitted in marginal/weak damping case!
- Propagation speed $c \sim \sqrt{D\gamma}$ (depends on α), characteristic scale $\ell \sim \sqrt{D/\gamma}$
- “Maxwell construction” for speed

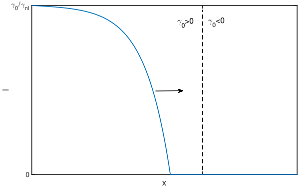
$$c \int_{-\infty}^{\infty} D(I(z)) I'(z)^2 \, dz = \int_0^1 D(I) f(I) \, dI$$

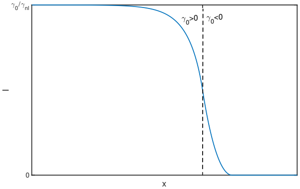
$z = x - ct$

- Thus turbulence spreads if $\alpha < \alpha^*$, recedes if $\alpha > \alpha^*$. Corresponds to (meta)stability of fixed points

Penetration into stable zone I

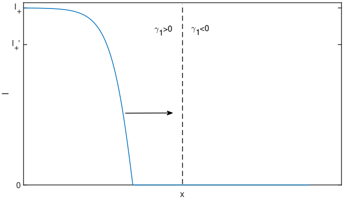
- Consider spreading of turbulence from lin. unstable to lin. stable zone
- Simple model: $\gamma_1 = \gamma_g > 0$ for $x < 0$, $\gamma_1 = -\gamma_d < 0$ for $x > 0$
- Allow turbulent front to form in lefthand region and propagate
- In Fisher model, penetration is *weak*: forms stationary, exponentially-decaying profile with $\lambda \sim \sqrt{D_0/\gamma_{nl}} \sim \Delta_c$. Dubiously consistent with observation

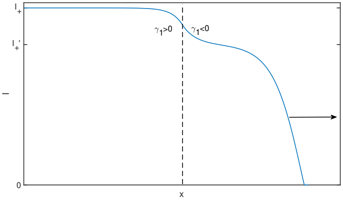




Penetration into stable zone II

- However, in our model, a new front with reduced speed/amplitude forms in second region if weakly damped (i.e. γ_d is small enough that $\alpha < \alpha^*$)
- Hence: can have ballistic propagation even in stable zone!
- More strongly delocalizing effect on the flux-gradient relation, compared to Fisher





Penetration into stable zone III

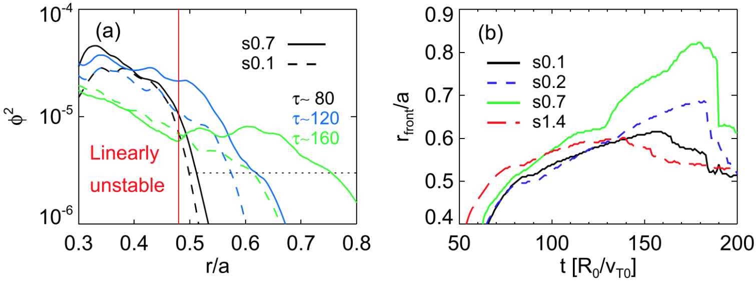
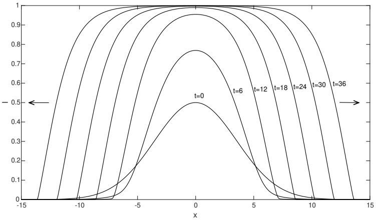
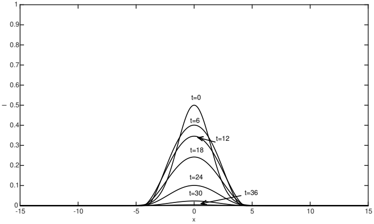


Figure: Spreading into stable zone in GK simulation with magnetic shear [Yi et al., 2014]. Ballistic propagation???

Local threshold behavior

- In contrast to Fisher, sufficiently large localized puff of turbulence will grow into front and spread. Suggestive of an avalanche triggered by initial seed
- How to determine threshold?

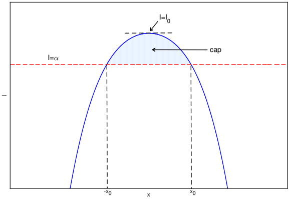




Two puffs differing only in spatial size are initialized; one grows and spreads, other collapses

Avalanche threshold

- Obviously puff amplitude must exceed $I_0 = \alpha$ or else $\gamma_{eff} = (I - \alpha)(1 - I) < 0$
- Consider "cap" of puff (part exceeding $I = \alpha$)
- Size threshold governed by competition between diffusion of turbulence out of cap and total nonlinear growth in cap (suggested by free energy functional)
- Sets scale $\sqrt{D/\gamma}$



Avalanche threshold (details)

- Strategy: assume initial puff is symmetric, has single max I_0 and single lengthscale L
- Expand intensity curve about max to quadratic order, plug into dynamical equation, integrate over extent of cap
- Result: growth if

$$L > L_{\min} = \sqrt{\frac{D(\alpha)I_0}{f(I_0) - \frac{1}{3}(I_0 - \alpha)f'(I_0)}} = \sqrt{\frac{3D\alpha I_0}{\gamma(I_0 - \alpha)((1 - 2\alpha)I_0 + \alpha)}}$$

- Power law $L_{min} \sim (I_0 - \alpha)^{-1/2}$

Avalanche threshold: analytical vs. simulation

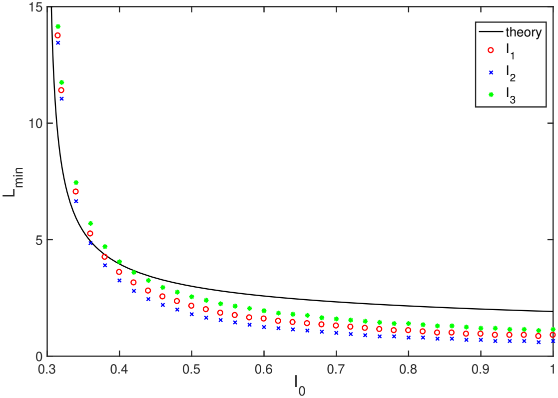


Figure: Numerical result for threshold at $\alpha = 0.3$ for three types of initial condition (Gaussian (I_1), Lorentzian (I_2), parabola (I_3)), compared with analytical estimate

Triggering an avalanche

- How might a puff of sufficient size form?
- Near linear marginality, threshold is weak:

$$I_{-} \sim \frac{|\gamma_1|}{\gamma_2} \ll 1, \; L_{\rm min} \sim \left(\frac{\chi_{GB}}{\omega_*}\right)^{1/2} \sim \Delta_c$$

- Suggests threshold can be triggered by noise
- Simulations of model with appropriate choice of noise (multiplicative + small additive background) show that front propagation events will be intermittently excited

Bistable model: conclusions

- Natural extension of Fisher model that allows for coexistence of laminar/turbulent domains
- Supported by substantial evidence for subcritical turbulence
- Provides simple framework for understanding avalanching: local exceedance of nonlinear instability by turbulent puffs
- Key testable predictions: ballistic spreading into weakly linearly damped regions, power-law threshold for spreading of puffs

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A teaser: machine learning model?

Towards a complete model

- A realistic model should include coupling to zonal flow and pressure profile
- Start with Hasegawa-Wakatani:

$$\partial_t n + \{\phi, n\} = \alpha(\phi - n) + \text{diss.}$$

$$\partial_t \nabla_\perp^2 \phi + \{\phi, \nabla_\perp^2 \phi\} = \alpha(\phi - n) + \text{diss.}$$

with $\alpha = -\eta {\partial_z}^2$ the adiabatic operator representing parallel electron response

- Take zonal averages:

$$\partial_t \langle n \rangle + \partial_x \langle \tilde{n} \tilde{v}_x \rangle = \text{diss.}$$

$$\partial_t \langle \zeta \rangle + \partial_x \langle \tilde{\zeta} \tilde{v}_x \rangle = \text{diss.}$$

$$\partial_t \langle \varepsilon \rangle + \langle (\tilde{n} - \tilde{\zeta}) \tilde{v}_x \rangle \partial_x \langle n - \zeta \rangle + \partial_x \langle \varepsilon \tilde{v}_x \rangle = \text{diss.}$$

where $\zeta = \nabla_\perp^2 \phi$, $\varepsilon = \frac{1}{2}(\tilde{n} - \tilde{\zeta})^2$

Learning mean field theory

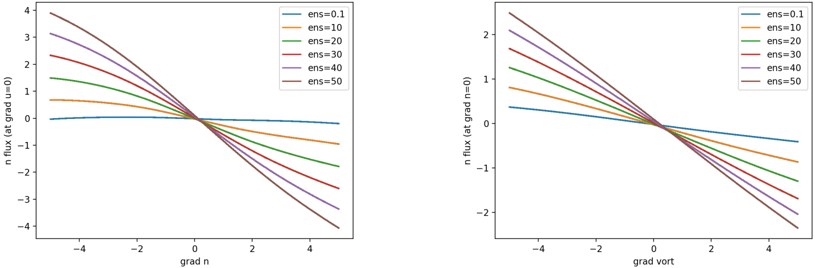
- How to proceed? Need model for turbulent fluxes $\Gamma_q = \langle \tilde{q}\tilde{v}_x \rangle$ but hard to calculate
- Idea: use simulations to train machine learning model that maps mean profiles to local fluxes
- Here ML is just a form of nonparametric regression: no need to impose a model
- One approach: local model

$$\Gamma_q(x) = f(\partial_x n|_x, \partial_x^2 n|_x, \ldots, \zeta|_x, \partial_x \zeta|_x, \ldots, \varepsilon|_x, \partial_x \varepsilon|_x, \ldots)$$

- Challenges: feature selection, noise suppression. Also is local model even valid?

Preliminary results: particle flux

- Training on $\sim$ 20 simulations of 2D Hasegawa-Wakatani at $\alpha = 2$ and constraining the model with symmetries of HW, a simple neural network learns a reasonable model for the particle flux



Learned turbulent particle flux as function of density gradient at zero vorticity gradient (left) and vice versa (right.)

Preliminary results: particle flux

- Flux is approximately linear combination of terms prop. to $\partial_x \langle n \rangle$ and $\partial_x \langle \zeta \rangle$. First is obvious, latter less so!
- No clear dependence on shear itself

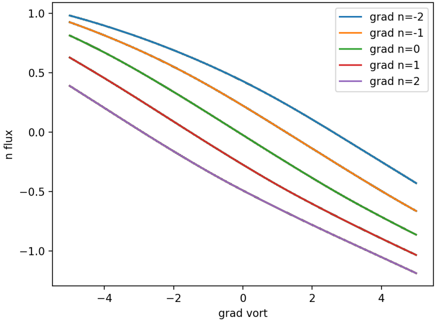


Figure: Dependence of particle flux on both density and vorticity gradients

Preliminary results: particle flux

- Results can be explained by simple quasilinear theory. However, must include effects of mean vorticity gradient on dispersion relation! Ignored in most studies

$$\partial_t \tilde{n} + V(x)\partial_y \tilde{n} + \partial_x \langle n \rangle \partial_y \tilde{\phi} = \alpha(\tilde{\phi} - \tilde{n})$$

$$\partial_t \tilde{\zeta} + V(x)\partial_y \tilde{\zeta} - V''(x)\partial_y \tilde{\phi} = \alpha(\tilde{\phi} - \tilde{n})$$

$$\omega = \frac{k_y}{1+k^2}(\partial_x \langle n \rangle + V'') + k_y V$$

  for $\alpha \gg 1$

- Real part of frequency proportional to PV gradient, not density gradient! This has lots of interesting consequences, just one of which is effect on particle flux

Preliminary results: particle flux

- Quasilinear flux in adiabatic limit can be computed as

$$\Gamma_n \simeq \frac{1}{\alpha}\sum_{\mathbf{k}} -\frac{k_y^2}{1+k^2}\left(k^2\kappa - V''\right)|\tilde{\phi}_{\mathbf{k}}|^2$$

- Good agreement with ML
- Vorticity gradient term can result in staircasing
- This project very much a work in progress (vorticity and enstrophy flux are harder), but this simple result shows its potential to elucidate new physics