A unified theory of zonal flows and corrugations in drift wave turbulence

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What's new ?

- An analysis including both zonal noise generation, modulational instability and their interaction based upon a systematic spectral closure for zonal shears and corrugations. Useful to distill simulation results into reduced models. Noise?

- Discovery of forward transfer of internal energy of density corrugations $\sim \left\langle \left| \overline{n}/n_0 \right|^2 \right\rangle$, which occurs along with the inverse transfer of kinetic energy.

  - Zonal flows may exhibit negative viscosity but corrugations do not (i.e., diffusivity is positive!).

- Realization of important implications of zonal cross correlations $\langle \overline{n} \nabla_\perp^2 \overline{\phi} \rangle$ which

  - appears in spectral transfer rates

  - govern the relative phasing of density corrugations and zonal shears

  - These can be correlated or anti-correlated depending on the sign of $\langle \overline{n} \nabla_\perp^2 \overline{\phi} \rangle$.

- Improvement of predator prey and L-H model dynamics in the light of the role of zonal noise, in seeding zonal flow formation.

Outline

- Introduction
- Motivation
- Spectral evolutions: Zonal auto-correlations and cross-correlations
- Feedback loop with zonal/polarization noise
- Noise effect on the L-H transition
- Conclusions and discussion

Introduction

- DW-ZFT has two components: drift waves(“wavy” - $k_\theta \neq 0$) and zonal modes ($k_\theta = k_z = 0$)

- Zonal modes are modes of minimal inertia, transport and damping.

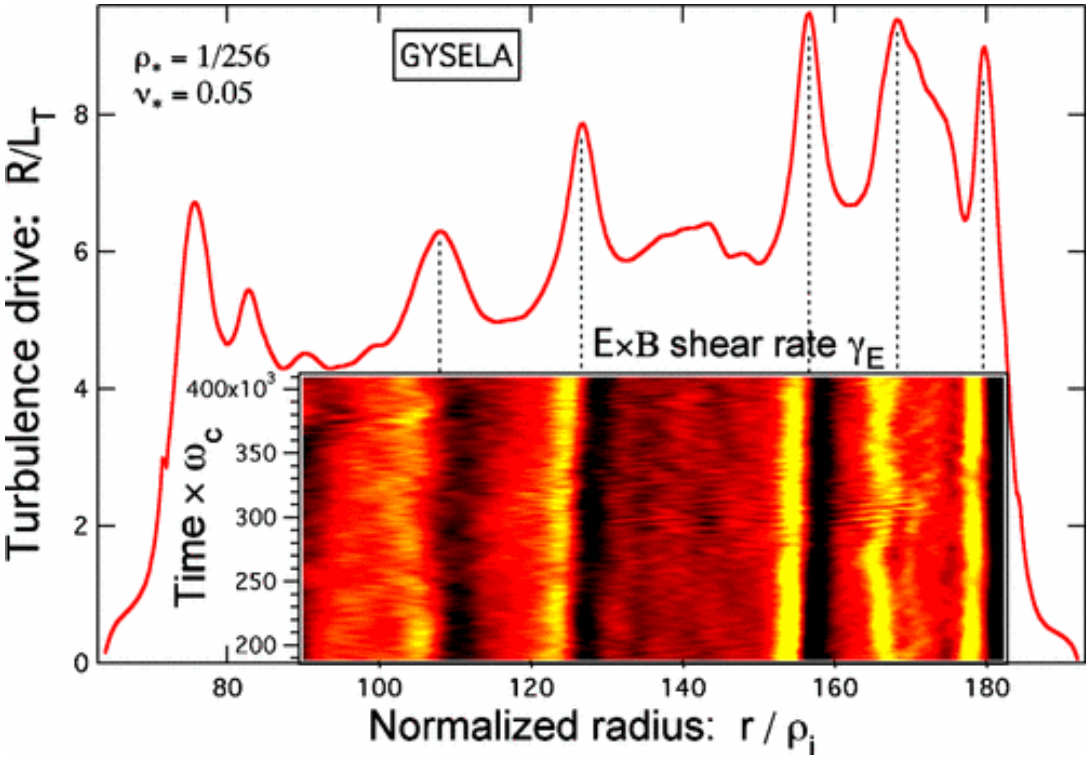
- Symmetry precludes adiabatic electron response for zonal modes. Thus they are benign repositories of fluctuation energy.

- Conversion of energy to zonal structures reduces transport and improves confinement.

- Zonal structures possible in

  different fields- ϕ,n,T_i,T_e…

Zonal flow   Profile corrugations



Motivation

- Almost all theoretical models of zonal flow generation divide cleanly into:

  1. Calculation of zonal flow dielectric or screening response, with occasional mention of wavy component beat noise. [RH 1998, HR, 1999].

$$\frac{\partial}{\partial t}\left\langle\left|\phi_q\right|^2\right\rangle = \frac{2\tau_c\left\langle\left|S_q\right|^2\right\rangle}{\left|\epsilon_{neo}\right|^2}$$

Emission from polarization interaction

- ignores modulational mechanism.

  2. Modulational stability calculations, which consider response of a pre-existing gas of drift waves to infinitesimal test shears or profile corrugations, but ignore noise emission.

$$\frac{\partial}{\partial t}\bar{\phi}_q = \int d\vec{k}\, k_y k_x \delta\left|\phi_k\right|^2 = -q_x^2\int d\vec{k}\, k_y^2 C_k \mathcal{R}^{(r)}_{k,q} k_x \frac{\partial\left\langle N_k\right\rangle}{\partial k_x}\bar{\phi}_q = \gamma_q\bar{\phi}_q$$

- What happens when the noise meets modulations? Langevin equation with -ve damping: $\frac{\partial\phi_q}{\partial t} - \gamma_q\phi_q = noise.$

- Adding noise in an unstable system is like adding gas to the fire. The whole picture with noise becomes a lot trickier. A unified theory of zonal modes is needed.

- This must be formulated at the level of coupled spectral equations, which treat incoherent noise emission and coherent response on equal footing.

Motivation

- There are both zonal flows and density corrugations at the simplest level of description of DW-ZFT.

- Zonal flows result from the inverse cascade of kinetic energy - this is well known. What about the density corrugations?

- How are the zonal density and zonal flow correlated ? → staircase?

- How do zonal flow and density corrugations feedback on turbulence?

  - Zonal flow shear induces diffusion of mean wave action density in k_x space . What happens to the turbulent kinetic energy and internal energy under the action of zonal shear?

  - How does density corrugation feedback on turbulence?

  - What are the implications of zonal noise on the predator prey dynamics?

  - How does zonal noise affect the dynamics of L-H transition?

DW-ZFT Spectral evolutions

To study zonal flow and density corrugations we use the simple Hasegawa-Wakatani model which has polarization NL $= \frac{1}{2} \sum_{\vec{p}+\vec{q}=\vec{k}} \hat{z} \cdot \vec{p} \times \vec{q} \left(q^2 - p^2\right) \phi_p \phi_q$ and advection NL $= \frac{1}{2} \sum_{\vec{p}+\vec{q}=\vec{k}} \hat{z} \cdot \vec{p} \times \vec{q} \left(\phi_p n_q - \phi_q n_p\right)$

- Adiabaticity parameter $\alpha \equiv \chi_e k_{\parallel}^2 / \omega_k$ is the main control parameter.

- Relevant quadratic spectra are kinetic energy spectrum $\left\langle \left| v_k \right|^2 \right\rangle = k^2 \left\langle \left| \phi \right|_k^2 \right\rangle$, internal energy spectrum $\left\langle \left| n_k \right|^2 \right\rangle$ and cross-correlation spectrum $\langle n_k \phi_k^\star \rangle$ - each for wavy and zonal modes.

- **Wavy cross-correlation spectrum** is related to flux $\Gamma_n = \sum_k - ik_y \langle n_k \phi_k^\star \rangle$. Wavy cross-correlation $\langle n_k \phi_k^\star \rangle$ is measure of alignment of fluctuations in DW turbulence - much like cross-helicity $\left\langle \vec{v} \cdot \vec{B} \right\rangle$ measures alignment in MHD turbulence. $\langle n_k \phi_k^\star \rangle$ high $\Longrightarrow \Gamma_n$ low, $\langle n_k \phi_k^\star \rangle$ low $\Longrightarrow \Gamma_n$ high.

- **Zonal cross-correlation spectrum** → alignment of zonal density and potential, relevant for staircase structure. Feedback on turbulence?

- Equation for spectral intensity involves calculation for triple correlation $\left\langle \phi_k^\star \phi_p \phi_q \right\rangle$ -DIA closure .

Zonal flow nonlocal straining of nonlinear invariants

Expanding about $\overrightarrow{p} = \overrightarrow{k}$ and retaining terms up to $\mathcal{O}(q_x^4)$ the spectral KE transfer function can be reduced to induced diffusion - thin isosceles triangles

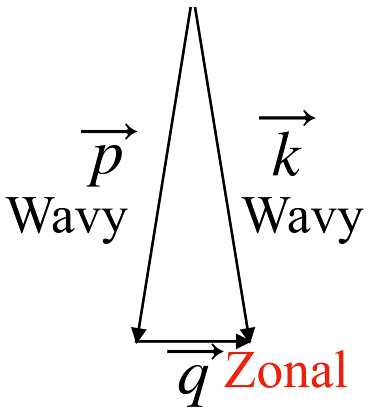
Wavy KE

Wavy cross-correlation

$$T_{\phi k} = \frac{\partial}{\partial k_x}\left[\sum_q 4k_y^2\left(\frac{k}{k_\perp}\right)^4 q_x^2 E_q \Theta_{kkq}^{(r)}\left(a_k^{Er}\frac{\partial E_k}{\partial k_x} - \frac{\partial a_k^{Er}}{\partial k_x}E_k\right)\right] + \frac{1}{k^2}\frac{\partial}{\partial k_x}\left[\sum_q 2k_y^2\left(\frac{k}{k_\perp}\right)^4 q_x^2 E_q \Theta_{kkq}^{(r)}\left(b_k^E\frac{\partial}{\partial k_x}k^2\left\langle n_k\phi_k^\star\right\rangle - \frac{\partial b_k^E}{\partial k_x}k^2\left\langle n_k\phi_k^\star\right\rangle\right)^r\right]$$

Zonal KE

Triad interaction time



Similarly for internal energy transfer function ….

- Spectral KE and IE are diffused and convected in k_x space by mean square zonal shear $q_x^4\left\langle\left|\phi_q\right|^2\right\rangle$.

- Diffusion of spectral KE and IE is coupled to diffusion of density-vorticity cross-spectrum.

- Sign of KE convection speed is +ve as $\dfrac{\partial a_k^{Er}}{\partial k_x}<0$.

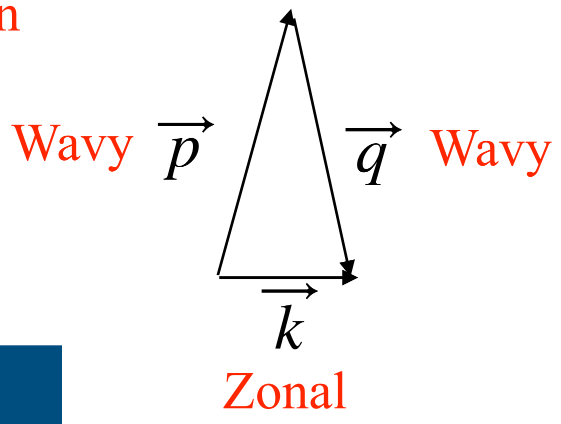
- $\Theta_{kkq}^{(r)}$ sets the strain coherence time.

Spectral evolution of zonal intensity

For the zonal mode $k_y = k_\parallel = 0$ and $k_x \neq 0$

Zonal cross-correlation

$$\left(\frac{\partial}{\partial t} + 2\mu k_x^2\right)\left\langle \left|\phi_k\right|^2 \right\rangle + 2\eta_{1k}^{zonal}\left\langle \left|\phi_k\right|^2 \right\rangle + \Re\left[2\eta_{2k}^{zonal}\boxed{\left\langle n_k \phi_k^\star \right\rangle}\right] = F_{\phi k}^{zonal}$$



Transfer Rates	Adiabatic regime	
Nonlinear damping	$\eta_{1k}^{zonal} = \sum_q k_x^2 q_y^2 \Theta^{(r)}_{k,-q,q} q_x \frac{\partial}{\partial q_x}\left[\left(1 - \frac{2q_\perp^4}{\left(1+q_\perp^2\right)^2}\frac{1}{\alpha_q^2}\right) I_q\right]$	-ve
Cross-transfer	$\eta_{2k}^{zonal} = -\sum_{\vec{q}} k_x^2 q_y^2 \Theta^{(r)}_{k,-q,q} q_x \frac{\partial}{\partial q_x}\left[\frac{I_q}{1+q^2}\right]$	+ve
Noise	$F_{\phi k}^{zonal} = 4\sum_q \Pi_q^2 \Theta^{(r)}_{k,-q,q}$; $\Pi_q = q_y q_x I_q$	+ve

- $\eta_{1k}^{zonal} \propto k_x^2$ and becomes -ve for $\frac{\partial I_q}{\partial q_x} < 0 \rightarrow$ transfer to large scales by **NEGATIVE VISCOSITY**

- Modulational instability when $-\eta_{1k} > \mu k_x^2$ defines a **critical spectral slope** !

- Zonal growth is. maximum when $\alpha_q \rightarrow \infty \Longrightarrow$ **Non-adiabatic fluctuations inhibit transfer to large scales !**

- $\eta_{2k}^{zonal,(r)} > 0$ ALWAYS for $\frac{\partial I_q}{\partial q_x} < 0 \Longrightarrow$ **Forward transfer when $\Re\left\langle n_k \phi_k^\star \right\rangle < 0$, backward transfer when $\Re\left\langle n_k \phi_k^\star \right\rangle > 0$** **NEW !**

Spectral evolution of density corrugations

$$\left(\frac{\partial}{\partial t}+2D_n k^2\right)\left\langle \left|n_k\right|^2\right\rangle+2\zeta_{1k}\left\langle \left|n_k\right|^2\right\rangle+\Re\left[2\zeta_{2k}\boxed{\left\langle n_k^\star \phi_k\right\rangle}\right]=F_{nk}$$

Zonal cross-correlation

NEW !

Corrugations damping rate: $\zeta_{1k}=\sum_{\vec{q}}\Theta^{(r)}_{k,-q,q}k_x^2q_y^2\frac{4q^4}{\left(1+q^2\right)^2}\frac{1}{\alpha_q^2}I_q\mathbf{>0}$

Cross-transfer rate: $\zeta_{2k}=\sum_{\vec{q}}\Theta^{(r)}_{k,-q,q}k_x^2q_y^2\frac{\left(1+2q^2\right)}{\left(1+q^2\right)^2}\frac{q^4}{\alpha_q^2}I_q\mathbf{>0}$**,** **Advection Noise:** $F_{nk}=4\sum_{\vec{q}}\Theta^{(r)}_{k,-q,q}k_x^2q_y^2\frac{q_\perp^4}{\alpha_q^2}I_q^2\mathbf{>0}$

- Density modulational damping ζ_{1k}, cross-coefficient ζ_{2k} and advection noise F_{nk} ALL +ve and scale as $\frac{1}{\alpha_q^2}$.
- Physics of the corrugations dynamics is different form the inverse cascade of zonal flows. Density cascade forward in k_x !
- Corrugations become weaker as the response become more adiabatic.
- Corrugation is determined by noise vs diffusion balance.
- This is important for the nonlinear dynamics underlying staircases. Forward cascade in k-space is supporting the idea of (inhomogeneous) mixing in real space.

Spectral evolution of zonal cross-correlation

From zonal vorticity and zonal density equation one can obtain **NEW!**

$$\frac{\partial}{\partial t}\left\langle \bar{n}\nabla_x^2\bar{\phi}\right\rangle - \left(\mu + D_n\right)\left\langle \nabla_x^2\bar{n}\nabla_x^2\bar{\phi}\right\rangle = \left\langle \Gamma_{nx}\nabla_x^3\bar{\phi}\right\rangle + \left\langle \nabla_x\Pi_{xy}\nabla_x\bar{n}\right\rangle$$

- $\Longrightarrow$Zonal correlations are determined by correlation of fluxes and profiles. Zonal correlations are relevant to spatial structure of profile.

- Significant for layering or staircase structure - potential and density are aligned in staircase!

Q: When do zonal density and zonal potential align?

From spectral closure calculations, in steady state

$$\Re\left\langle n_k\phi_k^\star\right\rangle = \frac{2\eta_{2k}^{(r)}\left\langle \left|n_k\right|^2\right\rangle + 2\zeta_{2k}^{(r)}\left\langle \left|\phi_k\right|^2\right\rangle}{-\left(\mu + D_n\right)k_x^2 - 2\xi_{1k}^{(r)}} = \begin{cases} +ve \quad when \ -\left(\mu + D_n\right)k_x^2 - 2\xi_{1k}^{(r)} > 0 \\ -ve \quad when \ -\left(\mu + D_n\right)k_x^2 - 2\xi_{1k}^{(r)} < 0 \end{cases}$$

Where $\xi_{1k}^{(r)} = \eta_{1k} + \zeta_{1k}$= non-lin zonal damping rate + non-lin corrugation damping rate

- $\Longrightarrow$Zonal density and potential are correlated (anti-correlated) when the modulational growth of zonal flow more (less) than modulational damping of corrugations.

Summary of zonal flow and corrugations interaction

(a) Zonal flow - Vorticity equation - Polarization charge flux		
Process	**Impact**	**Key physics**
Polarization noise	Seeds zonal flow	Polarization flux correlation, +ve definite
Zonal flow response (comparable to noise)	Drives zonal shear using DW energy	Non-local inverse transfer in k_x , -ve viscosity
Zonal shear straining of small scale	Regulates waves via straining	Stochastic refraction straining waves, induced diffusion to high k_x
(b) Density corrugations - Density equation - Particle flux		
Density advection beat noise	Seeds density corrugation	Advection beats due to non-adiabatic electrons.
Density corrugations response	Damps and regulates density corrugations	Non-local forward transfer in k_x +ve diffusivity, turbulent mixing weak for $\alpha >> 1$
Zonal shear straining of small scale	Regulates waves via straining	Stochastic refraction straining waves, induced diffusion to high k_x
(c) Zonal cross-correlation - Vorticity and density transport processes		
ZCC response	Sets corrugation - shear layer correlation; staircase states	Growth of zonal intensity must exceed the modulational damping rate of corrugation

Feedback loop with nonlinear zonal noise

How does zonal noise affect the feedback loops?

Turbulence energy ε evolves as

Induced diffusion

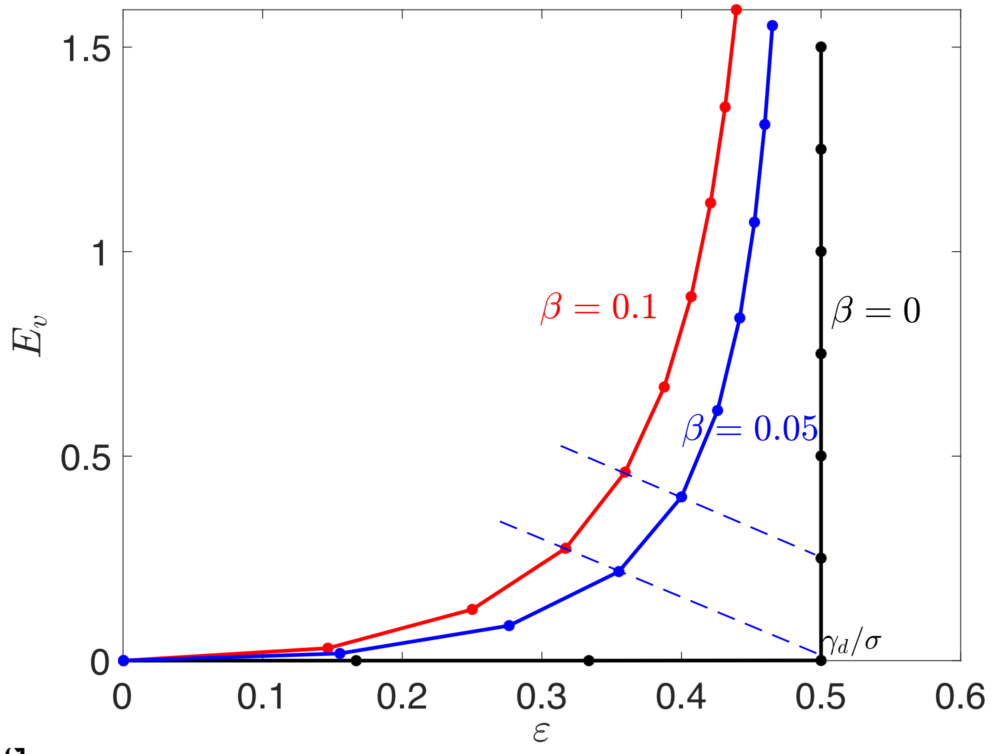
$$\frac{\partial \varepsilon}{\partial t} = \gamma\varepsilon - \sigma E_v \varepsilon - \eta\varepsilon^2$$

Nonlinear damping

Zonal flow energy E_v evolves as

Modulational growth

$$\frac{\partial E_v}{\partial t} = \sigma\varepsilon E_v - \gamma_d E_v + \beta\varepsilon^2$$



Without noise:

- Threshold in growth rate $\gamma > \eta\gamma_d/\sigma$ for appearance of stable zonal flows.
- Turbulence energy increases as γ/η below the threshold, until it locks at γ_d/σ, at the threshold.
- Beyond the threshold, turbulence energy remains locked at $\frac{\gamma_d}{\sigma}$ while the zonal flow energy continues to grow as $\sigma^{-1}\eta\left(\gamma/\eta - \gamma_d/\sigma\right)$.

With noise:

- **Both zonal and turbulence co-exist at any growth rate: - No threshold in growth rate for zonal flow excitation. Both zonal flow and turbulence energy increases with growth rate.**
- **Zonal flow energy is related to turbulence energy as $E_v = \beta\varepsilon^2/\left(\gamma_d - \sigma\varepsilon\right)$.**
- **Turbulence energy never hits the modulational instability threshold, absent noise!**
- **Turbulence energy decreases and zonal flow energy increases:- Noise feeds energy into zonal flow!**

Noise effect on the L - H transition

How does zonal noise affect the dynamics of L-H transition ?

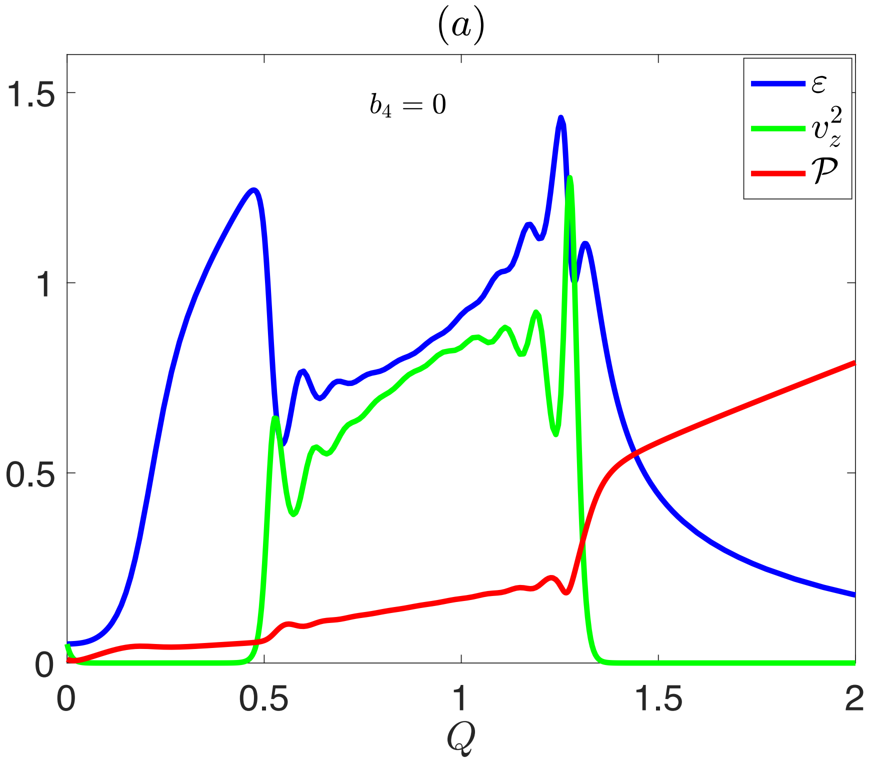
Extended KD 03 model with zonal noise [Minimal model]

Turbulence intensity: $$\frac{\partial \varepsilon}{\partial t} = \frac{a_1 \mathscr{P} \varepsilon}{1 + a_3 \mathscr{V}^2} - a_2 \varepsilon^2 - \frac{a_4 v_z^2 \varepsilon}{1 + b_2 \mathscr{V}^2}$$

Zonal flow energy: $$\frac{\partial v_z^2}{\partial t} = \frac{b_1 \varepsilon v_z^2}{1 + b_2 \mathscr{V}^2} - b_3 v_z^2 + b_4 \varepsilon^2$$

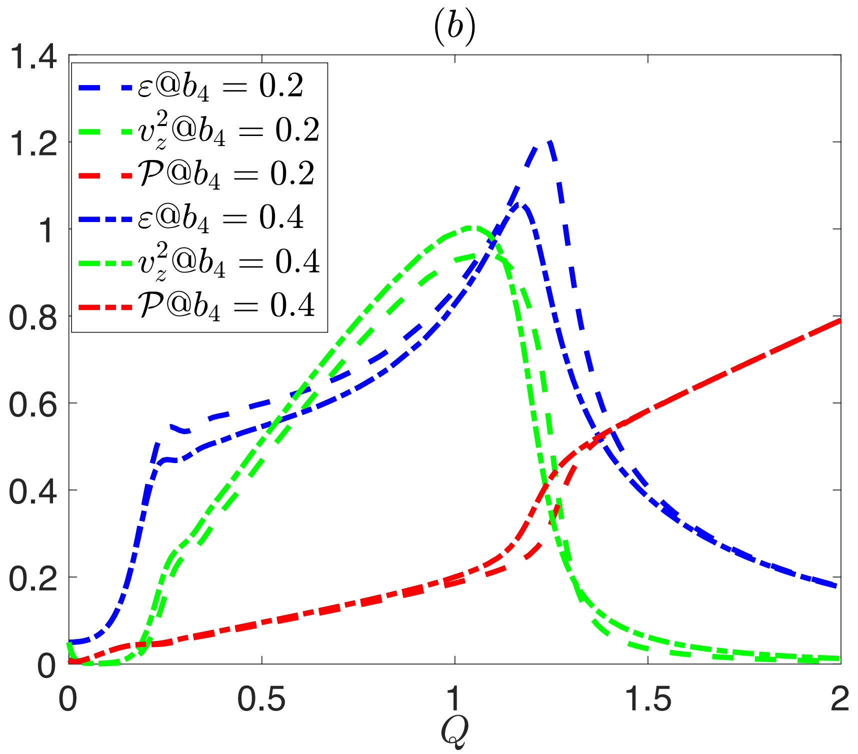
Pressure gradient: $$\frac{\partial \mathscr{P}}{\partial t} = -c_1 \frac{\varepsilon \mathscr{P}}{1 + c_2 \mathscr{V}^2} - c_3 \mathscr{P} + Q$$

Radial force balance: $\mathscr{V} = -\mathscr{P}^2$



Without noise:

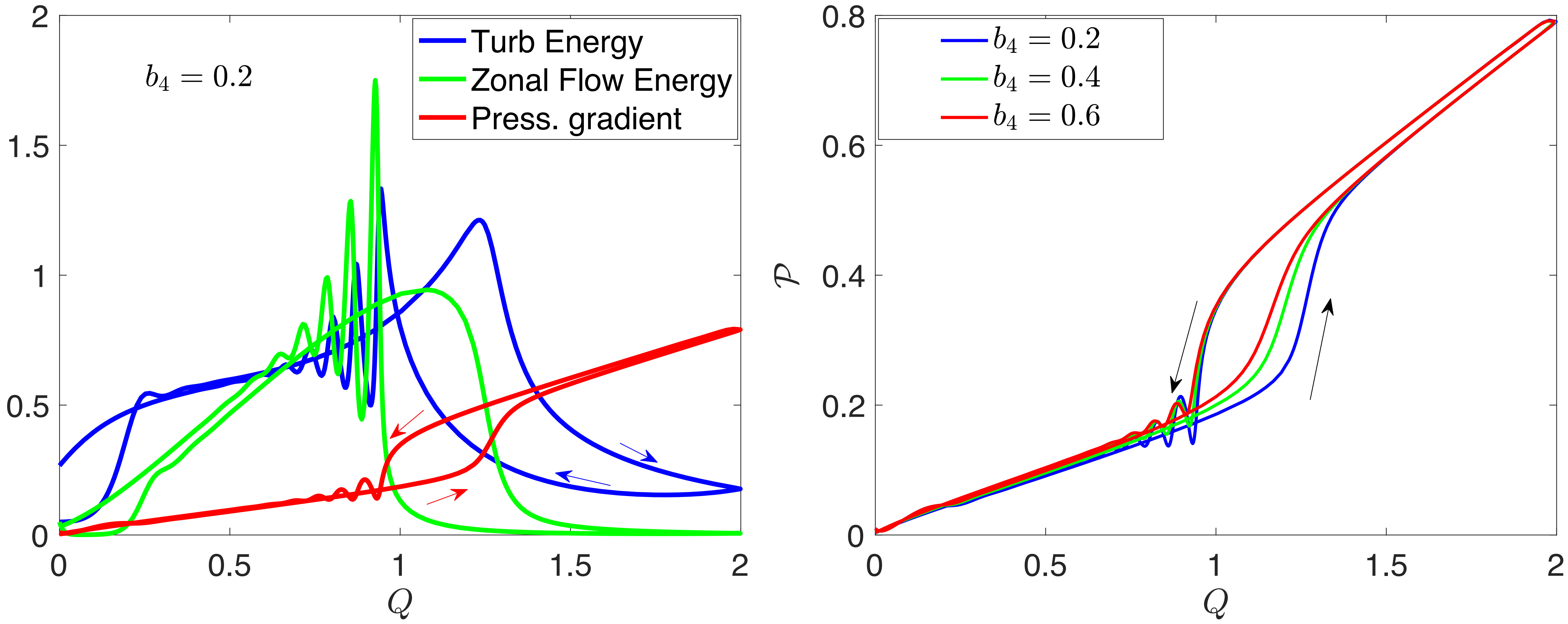
- Zonal flow appears only after a critical power, set by modulational instability threshold.
- Zonal flows exist only within the I-phase. Residual turbulence exist in H mode.



With Noise:

- **Significant zonal flows appear much below the modulational instability threshold. No ZF threshold in Q! Zonal flows exist at all Q!**
- **Turbulence level is reduced, no overshoot, zonal flow enhanced.**
- **Amplitude of I-Phase oscillations reduced.**
- **H-mode power threshold is reduced. Noise couples more fluctuation energy to zonal flows.**

Noise effect on L-H-L hysteresis



- The I-phase in the back transition is more oscillatory than that in the forward transition.

- Hysteresis with noise is robust w.r.t the variations in the initial conditions and the power retreat point in the H mode. Hysteresis is not so robust without noise.

- Threshold power for both forward and backward transition decreases with noise such that the area enclosed the hysteresis curve decreases with noise.

Conclusions I

We presented a unified theory of zonal mode dynamics. Derived a unified set of spectral equations, encompassing nonlinear response, polarization and advection beat noise.

New theoretical results:

- Vorticity flux correlations drive zonal flow noise. Likewise, density flux correlations drive corrugation noise.

- While effective viscosity for zonal flows can go negative, the zonal diffusivity remains positive for $\alpha > 1$. Bi-directional transfer- KE energy to large scales with internal energy to small scales.

- The effective zonal viscosity goes negative only for an energy spectrum which decays sufficiently rapidly in k_x i.e., $\partial E/\partial k_r < 0$ and $\left|\partial E/\partial k_r\right| < \left|\partial E/\partial k_r\right|_{crit}$. Importance of zonal cross-correlation $Z_{cc} \equiv \left\langle \bar{n} \nabla^2 \bar{\phi} \right\rangle$ identified. Z_{cc} determine the phasing of density corrugations and shear layers. Z_{cc}>0 when modulational growth of zonal shear exceeds the damping of density corrugations. $Z_{cc} \rightarrow 0$ implies no coherence of zonal patterns.

Conclusions II

Implications:

- Polarization beat noise and modulational effects are comparable intrinsically (both driven by Reynolds stress!). The synergy of the two mechanisms is stronger than either alone. This is because zonal noise acts to excite marginally stable and weakly damped zonal flows.
    - Expands the range of zonal flow activity relative to that predicted by modulational instability calculations.
    - Increases branching ratio of zonal flow energy to turbulence energy.
- Interaction of zonal noise and modulation has significant effect on feedback processes and thus the global characteristics of DW-ZFT.
    - Regarding the L-H transition: Noise eliminates the threshold for zonal flow excitation, and so expands the predicted range of the intermediate phase, drastically reduces the turbulence overshoot.
    - Answers: if zonal flows are the L-H trigger, then what triggers the trigger? → Polarization beat noise triggers the trigger!
    - The energy transfer to zonal flow is accelerated which lowers the threshold for L-H transition.

For experimentalists (Analog+Digital)

- Test the spectral transfer mechanism for corrugations → Bicoherence, etc.

- The zonal cross-correlation has not been measured and its relation to staircase structure has not been tested. Do so !

- Predator prey dynamics drastically changes when zonal noise is accounted for. The domain of zonal flow excitation expands, and the system never reached the modulational instability threshold. Test !

- The improved L-H transition model presented in this paper is testable. In particular, the weak overshoot, expanded domain of zonal mode activity, absence of a modulation instability and the level of residual H mode turbulence are all more consistent with experimental results than the results of earlier reduced models. Quantitative study?

N B: Well known that zonal flows appear before the I-phase. $\Longrightarrow$ Noise !

Future directions

- Deeper understanding of zonal flow generation :
  - Does shearing occurs in an intermittent and bursty avalanche - like feedback events? PDFs?
  - Does a critical spectral slope self-organize from these interactions?
- Understanding interaction of corrugations with avalanches:
  - Corrugations in state of high Z_{cc} sustained as localized transport barriers, staircases etc. localized by accompanying shear flow?
  - Corrugations in state of low Z_{cc} likely to overturn, and drive avalanches, as in running sandpile?
  - Relevant for TEM turbulence. Does the density gradient state consist of standing corrugations , running avalanches or mixtures thereof ?
- Theory should better understand the effect of noise on staircase, which have been considered only in context of Mean Field theory.
- Relation between Z_{cc} and the staircase structure: Does the physics of Z_{cc} set the relative positions of corrugations and shear layer? Is there a single Z_{cc} for staircase state ? Or a band ?