Role of cross-helicity in β-plane MHD turbulence

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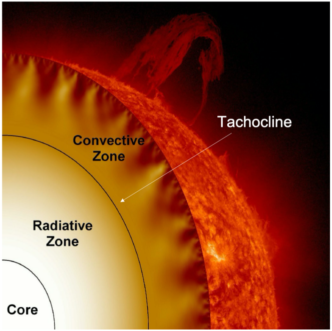
The speaker

Our speaker, Maya, could not be here today because she is a cat

Solar tachocline

- Thin, radially-sheared layer at base of convection zone. Strongly turbulent
- Believed to be strongly involved in the solar dynamo — "interface dynamo" [Parker (1993)]:
    1. differential rotation drags poloidal field lines originating from core, converts to strong toroidal field (Ω-effect)
    2. small-scale helical motion twists toroidal field into poloidal field, completing the loop (α-effect)
- Strong stratification in tachocline $\Longrightarrow$ quasi-2D



β-plane MHD model

- 2D magnetized incompressible turbulence in presence of planetary vorticity (Coriolis force) gradient: $2\boldsymbol{\Omega} = (0, 0, f + \beta y)$
- Serves as model for tachocline

$$\partial_t \nabla^2 \psi + \beta \partial_x \psi = \{\psi, \nabla^2 \psi\} - \{A, \nabla^2 A\} + \nu \nabla^4 \phi + \tilde{f}$$
$$\partial_t A = \{\psi, A\} + \eta \nabla^2 A + \tilde{g}$$

- $\mathbf{v} = (\partial_y \psi, -\partial_x \psi, 0)$, $\mathbf{B} = (\partial_y A, -\partial_x A, 0)$
- $\{a, b\} = \partial_x a \partial_y b - \partial_y a \partial_x b$
- In this work, $\tilde{g} = 0$

Effect of (weak) mean field

- Tobias *et al.* (2007) assessed impact of weak mean field $b_0\hat{x}$ on zonal flow formation
- Above a critical b_0, turbulence is "Alfvénized." Reynolds-Maxwell stress $\langle\partial_x\psi\partial_y\psi\rangle - \langle\partial_x A\partial_y A\rangle \sim \sum_{\mathbf{k}}(|\mathbf{v_k}|^2 - |\mathbf{B_k}|^2)$ small $\implies$ no ZF
- η large enough $\implies$ quenches magnetic turbulence $\implies$ critical b_0 can be quite large

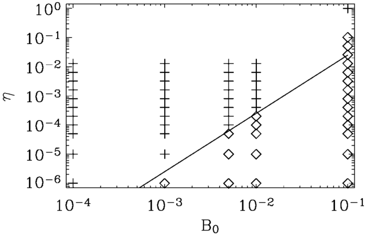


FIG. 5.—Scaling law for the transition between forward cascades (*diamonds*) and inverse cascades (*plus signs*). The line is given by B_0^2/η = constant.

Cross-helicity

- Previous analytical studies have neglected the effect of cross-helicity $\langle \mathbf{v} \cdot \mathbf{B} \rangle = -\langle A \nabla^2 \psi \rangle$. Often frozen at zero for simplicity, invoking usual conservation law
- However, Coriolis term explicitly breaks conservation:

$$\partial_t \langle A \nabla^2 \psi \rangle = -\beta \langle v_y A \rangle + \mathrm{dissipation}$$

- In this work: seek to elucidate the role of cross-helicity in this system. What is role in transport, ZF formation?

Stationary value

As a start, can obtain stationary CH value from a simple calculation à la Zeldovich. Neglecting forcing:

$$\frac{1}{2}\partial_t \langle A^2 \rangle = b_0 \langle A \partial_x \psi \rangle - \eta \langle (\nabla A)^2 \rangle$$

$$\implies \langle A \partial_x \psi \rangle_\infty = \frac{\eta}{b_0} \langle \tilde{b}^2 \rangle$$

$$\partial_t \langle A \nabla^2 \psi \rangle = -\beta \langle A \partial_x \psi \rangle + (\eta + \nu) \langle \nabla^2 \psi \nabla^2 A \rangle$$

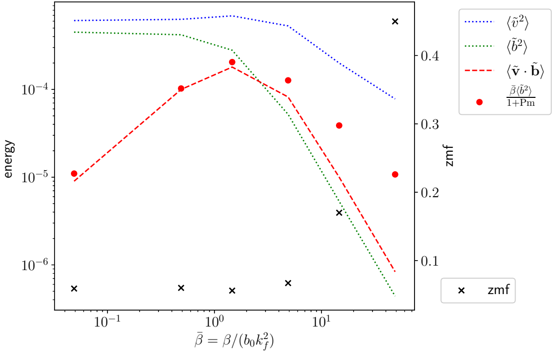
$$\implies \boxed{\langle A \nabla^2 \psi \rangle_\infty \simeq \frac{\beta \langle \tilde{b}^2 \rangle \ell_b \ell_v}{b_0 (1 + \mathrm{Pm})}}$$

where $\mathrm{Pm} \equiv \frac{\nu}{\eta}$

Note appearance of "magnetic Rhines" scale $k_{MR} = \sqrt{\frac{\beta}{b_0}}$, defines crossover of Rossby and Alfvén frequencies

Simulation results

- Simulate β-plane system with fixed $b_0 = 2$, $\eta = \nu = 0.01$, $\epsilon = 0.01$, $k_f = 32$ at various β
- Transition to Rossby turb. begins around $k_{MR} = k_f$ ($\beta = b_0 k_f^2$)
- Good agreement with Zeldovich with $\ell = \ell_f$ (breaks down for large β as $\ell_b < \ell_f$)
- Transition presaged by increasing mean CH — suggests CH plays a role?!



Spectra I

- Zeldovich calculation only yields large-scale mean — says nothing about transport. We need to look at spectra
- For tractability, assume $\mathrm{Pm} = 1$, use **weak wave turbulence theory**: resonant interactions between linear Rossby-Alfvén modes $\omega^{\pm} = \frac{1}{2}(-\omega_\beta \pm \sqrt{\omega_\beta^2 + 4\omega_A^2})$ with $\omega_\beta = -k_x\beta/k^2$
- Applicable when linear frequency is large compared to nonlinear scrambling rate
- To assess affect of β, assume large b_0, after a long time turn on β adiabatically

Spectra II

WWT spectral equations for arbitrary number of scalar fields ϕ^α (in eigenbasis) can be derived straightforwardly:

$$\partial_t C_{\mathbf{k}}^{\alpha\alpha'} = \sum_{\mathbf{k}'+\mathbf{k}''=\mathbf{k}} \sum_{\beta\gamma} \Bigg[|M_{\mathbf{k},\mathbf{k}',\mathbf{k}''}^{\alpha\beta\gamma}|^2 C_{\mathbf{k}'}^{\beta\beta} C_{\mathbf{k}''}^{\gamma\gamma} \delta(\omega_{\mathbf{k}}^{\alpha} - \omega_{\mathbf{k}'}^{\beta} - \omega_{\mathbf{k}''}^{\gamma}) \delta_{\alpha\alpha'}$$

$$+ M_{\mathbf{k},\mathbf{k}',\mathbf{k}''}^{\alpha\beta\gamma} M_{\mathbf{k}',\mathbf{k},-\mathbf{k}''}^{\beta\alpha\gamma} C_{\mathbf{k}}^{\alpha\alpha'} C_{\mathbf{k}''}^{\gamma\gamma} \left(\pi\delta(\omega_{\mathbf{k}}^{\alpha} - \omega_{\mathbf{k}'}^{\beta} - \omega_{\mathbf{k}''}^{\gamma}) - i\mathcal{P} \frac{1}{\omega_{\mathbf{k}}^{\alpha} - \omega_{\mathbf{k}'}^{\beta} - \omega_{\mathbf{k}''}^{\gamma}} \right)$$

$$+ M_{\mathbf{k},\mathbf{k}',\mathbf{k}''}^{\alpha'\beta\gamma*} M_{\mathbf{k}',\mathbf{k},-\mathbf{k}''}^{\beta\alpha'\gamma*} C_{\mathbf{k}}^{\alpha\alpha'} C_{\mathbf{k}''}^{\gamma\gamma} \left(\pi\delta(\omega_{\mathbf{k}}^{\alpha'} - \omega_{\mathbf{k}'}^{\beta} - \omega_{\mathbf{k}''}^{\gamma}) + i\mathcal{P} \frac{1}{\omega_{\mathbf{k}}^{\alpha'} - \omega_{\mathbf{k}'}^{\beta} - \omega_{\mathbf{k}''}^{\gamma}} \right) \Bigg].$$

where $\langle \phi_{\mathbf{k}}^{\alpha} \phi_{\mathbf{k}'}^{\alpha'} \rangle = C^{\alpha\alpha'} \delta(\mathbf{k} + \mathbf{k}') e^{-i(\omega_{\mathbf{k}}^{\alpha} - \omega_{\mathbf{k}}^{\alpha'})t}$, $M_{\mathbf{k}\mathbf{k}'\mathbf{k}''}^{\alpha\beta\gamma}$ are symmetrized nonlinear coupling coefficients. PV integrals vanish in case of real coupling coefficients and a single field, recover Sagdeev-Galeev result.

Spectra III

- Can compute exact WWT collision integrals for general b_0, β in principle by changing bases. In practice, very complicated
- Instead, compute correction to stationary spectrum to first order in β (in spirit of MF electrodynamics)
- Elsässer basis convenient: write $\mathbf{z}^{\pm} = \mathbf{v} \pm \mathbf{b}$, $\langle \mathbf{z}^{\pm}_{\mathbf{k}} \cdot \mathbf{z}^{\pm}_{\mathbf{k}'} \rangle = E^{\pm}_{\mathbf{k}} \delta(\mathbf{k} + \mathbf{k}')$, $\langle \mathbf{z}^{+}_{\mathbf{k}} \cdot \mathbf{z}^{-}_{\mathbf{k}'} \rangle = P_{\mathbf{k}} \delta(\mathbf{k} + \mathbf{k}')$
- In 2D MHD, asymptotic WWT spectra with no CH are *flat* $E^{+}_{\mathbf{k}} = E^{-}_{\mathbf{k}} = C, P_{\mathbf{k}} = 0$ [Tronko *et al.* (2013)]
- **How does finite β alter this spectrum?**

Spectra IV

- First-order result is (to leading order in $1/k_{\max}$—ultraviolet cut-off)

$$E_{\mathbf{k}}^{\pm} \simeq C \left(1 \pm \frac{\pi \beta}{8 b_0 k_{\max}} \frac{k_y^2}{k_x^2} \delta(k_x) \right) .$$

- Elsässer imbalance equivalent to finite cross-helicity, so leading-order effect of β is to induce CH at large parallel lengthscales.
- $\Delta CH \sim \beta \langle E \rangle / (\Delta k_x^2 b_0)$— looks consistent (up to $O(1)$ factors) with Zeldovich calculation
- But: Elsässer alignment $P_{\mathbf{k}} = 0$ remains stable $\implies |\mathbf{v}_{\mathbf{k}}|^2 \simeq |\mathbf{b}_{\mathbf{k}}|^2$. No impact on Maxwell-Reynolds competition

What we've learned so far

- Cross helicity is non-conserved in β-plane MHD
- In presence of mean magnetic field, attains a finite stationary value
- At first order, effect of β on spectrum is to induce finite shift in cross-helicity at large parallel lengthscales
- May play role in transition from Alfvénic to Rossby turbulence

Next steps

- Need to go to second order in β to assess role of shift in transport, transition from Alfvénic to Rossby turbulence
- Also: our result is problematically singular as $k_x \to 0$. Reflects fact that $\omega \to 0$, WWT breaks down
- Resolution: need to include strong turbulence/resonance broadening effects. Replace $\pi\delta(\Delta\omega) - i\mathcal{P}/\Delta\omega \to 1/(i\omega + \gamma)$
- Use EDQNM damping rate $\gamma = \gamma_{NL} + \gamma_A$ with nonlinear scrambling effect

$$\gamma_{NL} \propto \left(\int_0^k dk' \, k'^2 E_{k'} \right)^{1/2}$$

and Alfvén effect

$$\gamma_A \propto k \left(\int_0^k dk' \, E^M_{k'} \right)^{1/2} .$$

Small scale RMS b-field relaxes triplet correlations in one Alfvén time [Pouquet (1978)]

Thank you for your attention!

Figure APS-DPP audience member paying close attention to talk on turbulence theory.