Thesis defense:
"Topics in mesoscopic turbulent transport"

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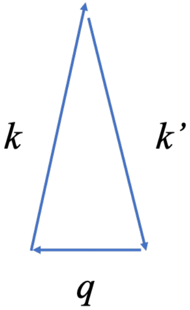
Introduction

Three projects on plasma turbulence. Unifying feature: interaction of turbulent microscales $\Longrightarrow$ meso-/macro-scale transport

1. Use machine learning to find reduced model for particle/momentum transport in drift-wave turbulence
2. New model for turbulence spreading and avalanching
3. Study relationship between cross-helicity and momentum transport in β-plane MHD



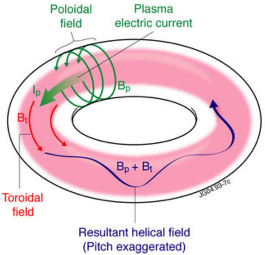
Outline

1. Background: drift wave turbulence
2. Deep learning project
3. Spreading project
4. Beta-plane MHD project
5. Extra slides

Background: drift wave turbulence

Tokamak physics basics

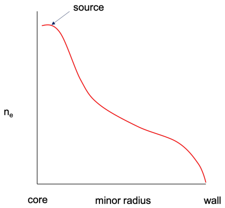
- Toroidal fusion device that uses strong helical magnetic field to confine plasma
- Key challenge: $\langle n\rangle\langle T\rangle\tau_E > 10^{21}\ \mathrm{keV\ s/m^3}$ (Lawson criterion) $\rightarrow$ maximize confinement time τ_E $\rightarrow$ minimize losses due to transport
- But: n, T gradients $\rightarrow$ instabilities $\rightarrow$ turbulence $\rightarrow$ anomalous transport. How to understand?



Drift waves

- Drift wave turbulence is useful paradigm for turbulence due to gradient instabilities (universal)
- Drift wave: collective oscillations associated with ion/electron diamagnetic drifts, which form in response to temperature/density gradients $v_d = 1/(qnB^2)\nabla p \times \mathbf{B}$
- Structure: cell convecting around $\tilde{n}$ at $v_E = -c/B^2\nabla\tilde{\phi} \times \mathbf{B}$, traveling at v_d

force balance $q(\mathbf{E} + \mathbf{v}_\perp \times \mathbf{B}) = \nabla p/n$



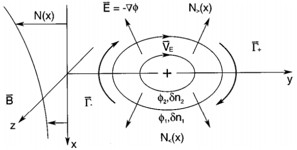
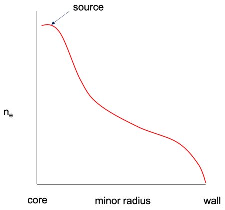


FIG. 1. Drift-wave mechanism showing $\mathbf{E}\times\mathbf{B}$ convection in a nonuniform, magnetized plasma.

Drift wave turbulence

- $\tilde{n}$ coupled tightly to $\tilde{\phi}$ by fast parallel "Boltzmann" electron response (from force balance $n_e e \partial_z \tilde{\phi} = T_e \partial_z n_e$) $n_e \simeq n_0 \exp(e\tilde{\phi}/T_e) \to \tilde{n}/n_0 \simeq e\tilde{\phi}/T_e$
- Collisions and resonances $\to$ phase shift $\tilde{n}_{\mathbf{k}}/n_0 \simeq e\tilde{\phi}_{\mathbf{k}}/T_e(1 - i\delta_{\mathbf{k}}) \to$ instability!
- Turbulence results when many drift modes unstable, nonlin. interaction becomes important



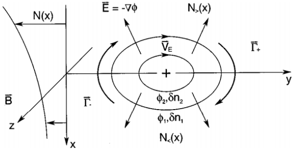


FIG. 1. Drift-wave mechanism showing **E**×**B** convection in a nonuniform, magnetized plasma.

Zonal flows

- Special modes with $m = n = 0$, $\omega \simeq 0$. Turbulence-driven, sheared poloidal flows
- In certain regime, spontaneously build up via secondary instability (multiscale interaction)
- No radial flow $\rightarrow$ do not cause harmful transport. "benign" free energy repository
- ZF shear stretches turbulent eddies $\rightarrow$ regulate turbulence
- Extremely important for confinement problem: zonal flows induce L-H transition

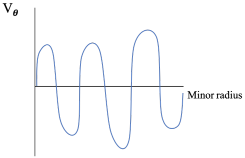


Figure ZFs also important in geophysical flows

Hasegawa-Wakatani model

- Simplest realistic framework for understanding collisional drift wave/zonal flow system.
- Coupled dynamics for potential ϕ, electron density n (dimensionless units):

$$\frac{dn}{dt} = \alpha(\phi - n) + D\nabla^2 n$$

$$\frac{d\nabla^2\phi}{dt} = \alpha(\phi - n) + \mu\nabla^4\phi$$

$$\frac{d}{dt} \equiv \frac{\partial}{\partial t} + (\hat{z} \times \nabla\phi)\cdot\nabla$$

- $\alpha \equiv k_\parallel^2 T_e/(n_0\eta\Omega_i e^2)$ "adiabaticity parameter," measures parallel electron response
- ϕ is stream function for flow $\mathbf{v}$

Deep learning project

Motivation: mean-field Hasegawa-Wakatani

- Want theory for radial transport
- Averaging over symmetry directions ($\langle \cdots \rangle$) yields

$$\partial_t \langle n \rangle + \partial_x \Gamma = \text{dissipation}$$

$$\partial_t \langle \nabla^2 \phi \rangle - \partial_x^2 \Pi = \text{dissipation}$$

$$\partial_t \varepsilon + 2\varepsilon(\Gamma - \partial_x \Pi)(\partial_x \langle n \rangle + \partial_x^3 \langle \phi \rangle) = -\gamma\varepsilon - \gamma_{NL}\varepsilon^2$$

  where $\Gamma = \langle \tilde{n} \tilde{v}_x \rangle$ (particle flux) and $\Pi = \langle \tilde{v}_x \tilde{v}_y \rangle$ (poloidal momentum flux or "Reynolds stress").
- $\varepsilon = \langle (\tilde{n} - \nabla^2 \tilde{\phi})^2 \rangle$ is turbulent potential enstrophy. Proxy for turbulence intensity
- Seek mean-field closure: Γ, Π as function of $\langle n \rangle$, $\langle \phi \rangle$, ε, radial derivatives. Idea: use supervised learning. Can we do better than simple QLT?

Feature selection: what do we want our model to look like?

- Assume a **local** model: local mean fields (in space and time) suffice to specify the local fluxes
- HW invariant under uniform shifts $n \to n + n_0$ and $\phi \to \phi + \phi_0 \implies$ eliminate dependence on $\langle n \rangle, \langle \phi \rangle$
- Invariant under poloidal boosts

$$\begin{cases} \phi & \to \phi + v_0 x \\ y & \to y - v_0 t \end{cases}$$

$\to$ eliminates dependence on ZF speed $V_y = -\partial_x \langle \phi \rangle$.

- Confine ourselves to adiabatic regime so $\tilde{n} \sim \tilde{\phi} \implies \varepsilon$ reasonably suffices to specify intensity
- Anticipate that hyperviscosity necessary to regularize ZF, so need derivatives up to V_y'''

Methods

- Thus choose minimal set of inputs $N', U, U', U'', \varepsilon$ $(N = \langle n \rangle, U = V_y')$
- 32 simulations of 2D HW, with $\alpha = 2$, various initial conditions for mean density, flow
- Postprocess to compute inputs, Γ, Π. Key: locality means each point in space, time treated on equal footing $\rightarrow$ lots of data per simulation
- Train neural network to output fluxes as functions of inputs
- Exploit/enforce 3 reflection symmetries via data augmentation

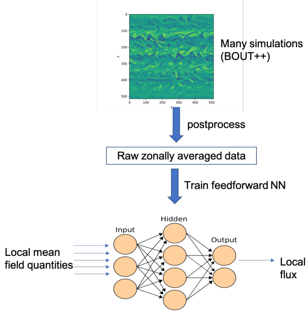


Figure Schematic of deep learning method

Neural networks 101

- **Bottom line: simply a proven form of nonparametric, multivariate regression**
- Use simplest form (multilayer perceptron)
- Inputs $\mathbf{x}$ repeatedly transformed $x_j^{(n+1)} = \sigma(W_{ij}^{(n)} x_i^{(n)} + b_j)$ where σ is a nonlinear function ("activation")
- Weights $\mathbf{W}^{(n)}$, biases $\mathbf{b}$ are "trained" using sophisticated algorithm to minimize loss function which measures deviation from labeled samples

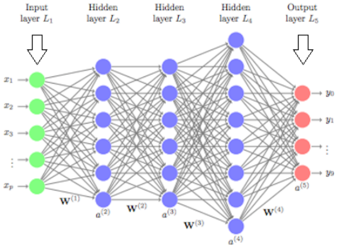


Figure Diagram of MLP, shamelessly stolen from the internet

Particle flux results

DNN learns a model roughly of the form (for small gradients)

$$\Gamma \simeq -D_n \varepsilon N' + D_U \varepsilon U'$$

Diffusive term $\propto N'$ is well-known, tends relax driving gradient. Second (non-diffusive) term not well-known, driven by vorticity gradient!

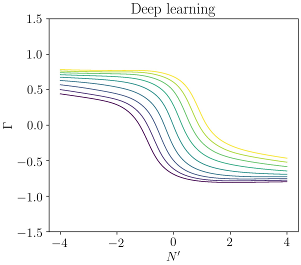


Figure Particle flux at constant ε as function of density and vorticity gradients

Derivation of nondiffusive term

$\alpha \to \infty$ calculation reproduces nondiffusive term. Need include frequency shift due to ZF! (quasilinear treatment, i.e. flux assumed due to coherent unstable drift waves)

$$\omega_{\mathbf{k}} = \frac{k_y}{1+k^2}(N' + U') + O(\alpha^{-2})$$

$$\gamma_{\mathbf{k}} = \frac{k_y^2}{\alpha(1+k^2)^3}(N' + U')(k^2 N' - U') + O(\alpha^{-2})$$

$$\Gamma = \mathrm{Re} \sum_{\mathbf{k}} -ik_y \tilde{n}_{\mathbf{k}} \tilde{\phi}_{\mathbf{k}}^*$$

$$= \sum_{\mathbf{k}} \frac{-k_y^2 \partial_x n(\gamma_{\mathbf{k}} + \alpha) + \alpha k_y \omega_{r,\mathbf{k}}}{\omega_{r,\mathbf{k}}^2 + (\gamma_{\mathbf{k}} + \alpha)^2} |\tilde{\phi}_{\mathbf{k}}|^2$$

$$= \frac{1}{\alpha} \sum_{\mathbf{k}} -\frac{k_y^2}{1+k^2} \left(k^2 N' - U'\right) |\tilde{\phi}_{\mathbf{k}}|^2 + O(\alpha^{-2})$$

Comparison to theory (diffusive term)

Compare DNN result to theory result using spectrum centered at most unstable $\mathbf{k}$ for $U' = 0$

$$\varepsilon_{\mathbf{k}} = \frac{\langle \varepsilon \rangle}{2\pi^2 \Delta k_x \Delta k_y} \frac{1}{1 + k_x^2/\Delta k_x^2} \left(\frac{1}{1 + (k_y - \sqrt{2})^2/\Delta k_y^2} + \frac{1}{1 + (k_y + \sqrt{2})^2/\Delta k_y^2} \right)$$

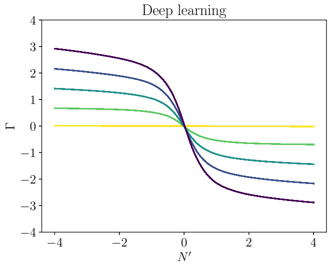


Figure Curves (at fixed $U = U' = U'' = 0$, and various ε) of Γ vs density gradient from DNN

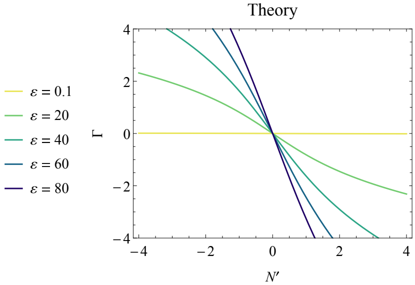


Figure Corresponding curves from QLT+ansatz with $\Delta k_x = \Delta k_y = 0.8$

Comparison to theory (nondiffusive term)

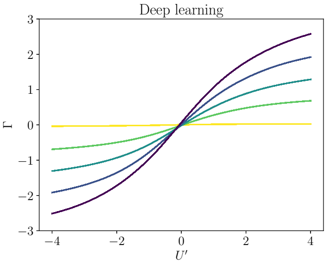


Figure Curves (at fixed $N' = U = U'' = 0$, and various ε) of Γ vs U' from DNN

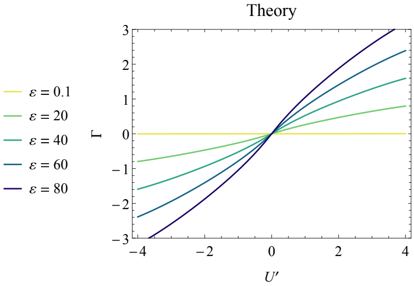


Figure Corresponding curves from QLT+ansatz with $\Delta k_x = \Delta k_y = 0.8$

Good agreement when $\partial_x n, \partial_x U$ are small!

Implications of nondiffusive term

- Neglected in literature, but coupling same order of magnitude (~ 0.5) that of usual N' term. Stronger than coupling to shear!
- Consequence: ZF can induce "staircase" pattern on profile. If $V_y = V_0 \sin(qx)$, U' term will contribute

$$\partial_t \langle n \rangle \sim -\frac{k_y^2 q^3 V_0 \langle \varepsilon \rangle}{\alpha (1+k^2)^3} \cos(qx)$$

- Previous explanation for staircase is some form of bistability. This mechanism is distinct.

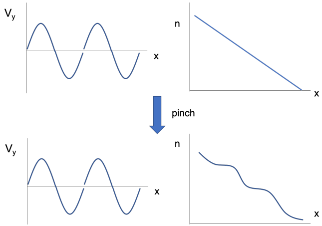


Figure Cartoon indicating how ZF may induce profile staircase via nondiffusive flux/pinch

Reynolds stress results

- Learns model of (Cahn-Hilliard) form (leading order)

$$\Pi = \varepsilon(-\chi_1 U + \chi_3 U^3 - \chi_4 \partial_x^2 U)$$

with $\chi_1, \chi_3, \chi_4 > 0$

- $\partial_t U = \partial_x^2 \Pi \sim \chi_1 \varepsilon k^2 U$. Zonal flow generation by *negative viscosity* $\varepsilon\chi_1$
- Large U stabilized by nonlinearity $\propto U^3$, small scales by hyperviscosity χ_4 (not shown)
- Agrees with previous theoretical models for zonal flow generation
- Recovery of hyperviscous is sensitive test of method

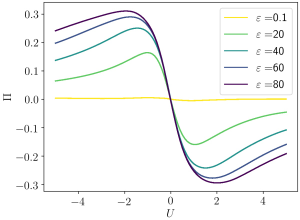


Figure Reynolds stress as function of U, at fixed U', U''

Reynolds stress: gradient corrections

- How does Reynolds stress depend on N', U'? Not easy to calculate
- Learned dependence well-described by overall suppression factor $f \simeq 1/(1+0.04(N'+4U')^2)$, i.e. gradients generally reduce Reynolds stress
- Found to be crucial for stability of learned model. Kinks tend to form in flow in its absence

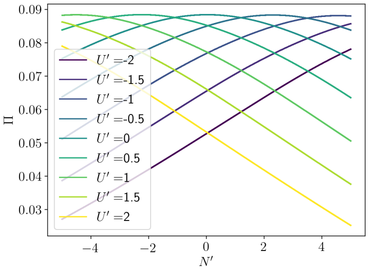
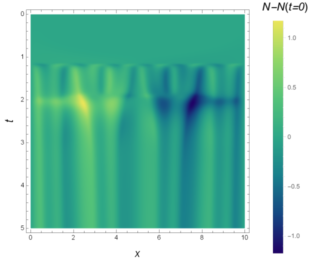
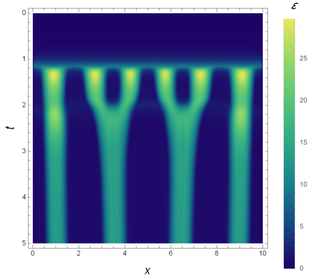
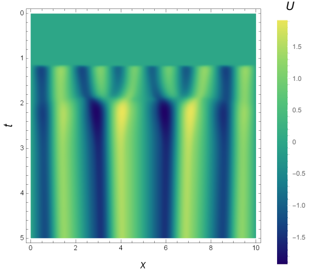


Figure Reynolds stress dependence on gradients at fixed ε, U, U''

Numerical solution of reduced 1-D model







Choose analytical expressions to match deep learning results, solve using implicit scheme

Conclusions

- ML recovers CH theory for ZF generation, while finding nontrivial gradient corrections
- Highlights rarely-discussed coupling of profile to flow, which induces profile layering
- Were confined to single adiabatic α, $N' \lesssim 3$. Otherwise, vortex interactions $\rightarrow$ 1D model doesn't make sense
- Test of concept for more complex applications. Geometry? 3D? T, B coupling?
- May need to relax some assumptions: multiple intensities? Spatial and/or temporal nonlocality?
- Tradeoff b/t complexity and interpretability
- Spreading???

Spreading project

Introduction

- Turbulence spreading = radial self-propagation of turbulence. Important in DWT
- Nonlinear coupling of microscales to mesoscopic envelope scale. Closure of $E \times B$ with envelope:

$$\partial_t \varepsilon_{\mathbf{k}} \sim -\sum_{\mathbf{k}'} (\mathbf{k} \cdot \mathbf{k}' \times \hat{z})^2 |\tilde{\phi}_{\mathbf{k}'}|^2 R(\mathbf{k}, \mathbf{k}') I_{\mathbf{k}} \to \frac{\partial}{\partial x} D_x(I_{\mathbf{k}}) \frac{\partial}{\partial x} I_{\mathbf{k}} - \mathbf{kk} : \mathbf{D} I_{\mathbf{k}}$$

$$D_x = \sum_{\mathbf{k}'} k_y'^2 |\phi_{\mathbf{k}'}|^2 R(\mathbf{k}, \mathbf{k}')$$

- Decouples flux-gradient relation: local turbulence intensity now depends on global properties of the profiles
- Fluctuations in linearly stable regions!

Depiction of spreading

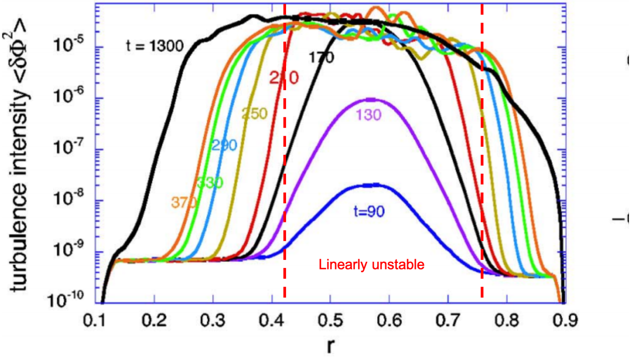


Figure Spatiotemporal evolution of flux-surface-averaged turbulence intensity in toroidal GK simulation. Linearly unstable region is $0.42 < r < 0.76$. From [Wang et al., 2006]

Avalanches

- Fast, intermittent transport events. Can account for a large percentage of total flux!
- Initially localized fluctuation cascades through neighboring cells via gradient coupling. Cell microscales couple with mesoscopic avalanche scale
- Associated with profile relaxation, SOC
- Closely related to spreading: both result in fast, mesoscopic turb front propagation. Unified model?

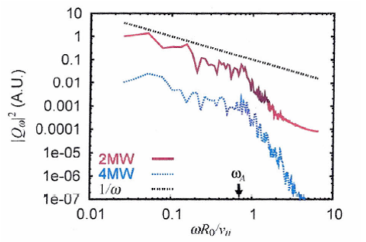


Figure Heat flux spectrum from GK simulation showing $1/f$ scaling

Fisher model

- Conventional wisdom for spreading is Fisher-type equation for turbulence intensity:

$$\partial_t I = \underbrace{\gamma_0 I}_{\substack{\text{local lin.} \\ \text{growth/decay}}} - \underbrace{\gamma_{nl} I^2}_{\substack{\text{local nonlin.} \\ \text{coupling to} \\ \text{dissipation}}} + \underbrace{\partial_x (D_0 I \partial_x I)}_{\text{nonlin. diffusion of turb. energy}}$$

- For $\gamma_0 > 0$, dynamics characterized by traveling fronts connecting unstable "laminar root" $I = 0$ and saturated "turbulent root" $I = \gamma_0/\gamma_{nl}$ with speed $c = \sqrt{\frac{D_0 \gamma_0^2}{2\gamma_{nl}}}$

Depiction of Fisher evolution

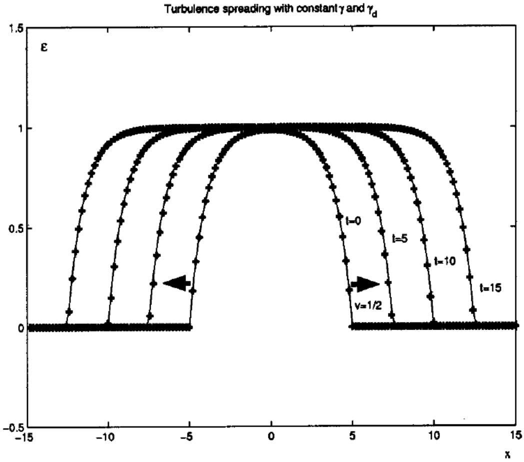


Figure Evolution of traveling turbulence front in Fisher model. From [Gürcan and Diamond, 2006]

Problems with Fisher

- Weak spreading into stable zone (few Δ_c). Dubiously consistent with experiment?
- If unstable, *why didn't noise already excite the whole system to turbulence?*
- Unless $\Delta x^2 \gamma_{nl} \ll D_0$, physical fronts require *bistability* à la [Pomeau, 1986]
- Growing body of evidence for bistable MF turbulence e.g. [Biskamp and Walter, 1985, Drake et al., 1995, Barnes et al., 2011, van Wyk et al., 2016]

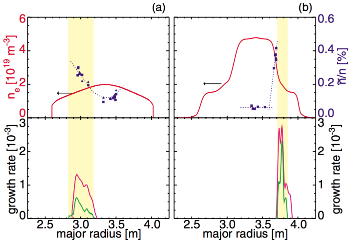


Figure Experiment by Nazikian et al 2005 clearly showing fluctuations in stable zone

Bistable model

- Propose phenomenological model of form

$$\partial_t I = \gamma_1 I + \gamma_2 I^2 - \gamma_3 I^3 + \partial_x (D(I) \partial_x I)$$

- take $D(I) = D_0 I$
- New physics: nonlinear turbulence drive $\propto I^2$. Can sustain sufficiently large fluctuations even when linearly damped
- *Bistable* in weak damping regime
- Estimate $\gamma_1 \sim \epsilon\omega_*,\ \gamma_{2,3} \sim \omega_*,\ D_0 \sim \chi_{GB}$

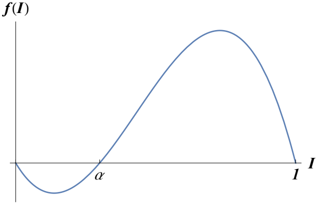
Model analysis I

$$\partial_t I = \gamma_1 I + \gamma_2 I^2 - \gamma_3 I^3 + \partial_x (D(I)\partial_x I)$$

- Qualitatively similar to Fisher EXCEPT in weak damping case $\gamma_1 < 0$ and $\gamma_2^2 > 4|\gamma_1|\gamma_3$
- Can then transform to Zel'dovich/Nagumo equation

$$\partial_t I = f(I) + \partial_x (DI\partial_x I)$$
$$f(I) \equiv \gamma I(I-\alpha)(1-I)$$

where $\alpha \equiv I_-/I_+,\ \gamma \equiv I_+^2\gamma_3,\ D \equiv I_+ D_0,\ I_\pm \equiv (\gamma_2 \pm \sqrt{\gamma_2^2 - 4|\gamma_1|\gamma_3})/2\gamma_3$



Model analysis II

- Unlike Fisher, traveling fronts admitted in weak damping case!
- Propagation speed $c \sim \sqrt{D\gamma}$ (depends on α), characteristic scale $\ell \sim \sqrt{D/\gamma}$
- "Maxwell construction" for speed

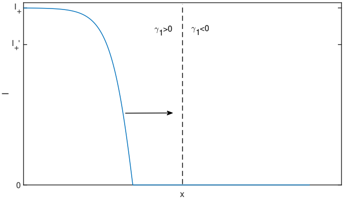
$$c \int_{-\infty}^{\infty} D(I(z)) I'(z)^2 \, dz = \int_0^1 D(I) f(I) \, dI$$

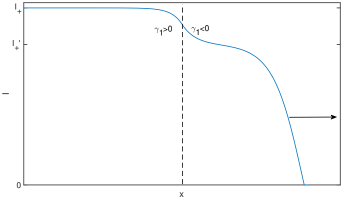
  $z = x - ct$

- Thus turbulence spreads if $\alpha < \alpha^*$, recedes if $\alpha > \alpha^*$. Also corresponds to (meta)stability of fixed points (Lyapunov functional)

Penetration into stable zone

- Fisher model: evanescent penetration, depth $\ell \sim \rho_s$
- Our model: new front with reduced speed/amplitude forms in second region if weakly damped (i.e. γ_d is small enough that $\alpha < \alpha^*$)
- Hence: can have ballistic propagation even in stable zone! **Much stronger penetration, delocalization**





Penetration into stable zone II

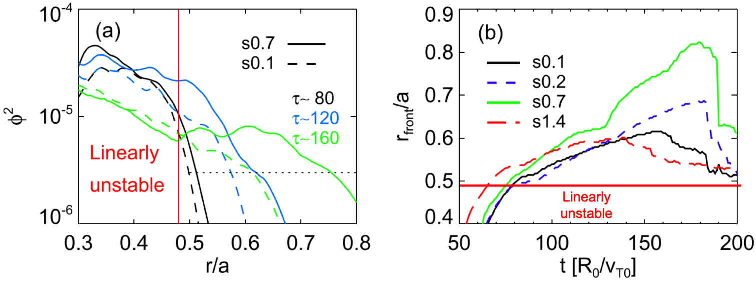
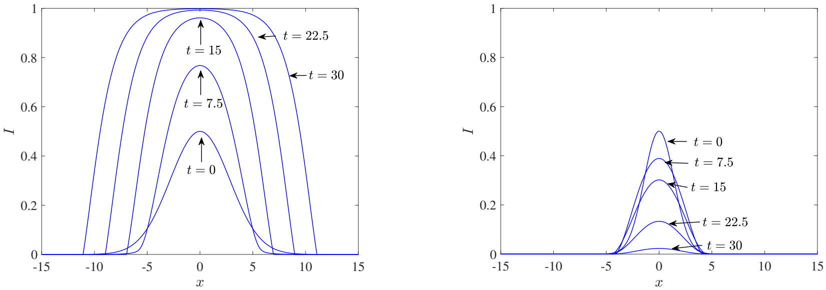


Figure Spreading into stable zone in GK simulation with magnetic shear [Yi et al., 2014]. Ballistic propagation???

Avalanche threshold I

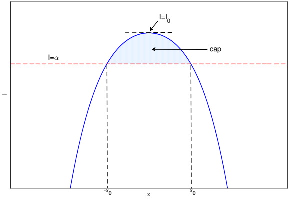
- In contrast to Fisher, sufficiently large localized puff of turbulence will grow into front and spread. Suggestive of an avalanche triggered by initial seed
- How to determine threshold?



Two puffs differing only in spatial size are initialized; one grows and spreads, other collapses

Avalanche threshold II

- Obviously puff amplitude must exceed $I_0 = \alpha$ or else $\gamma_{eff} = (I - \alpha)(1 - I) < 0$
- Consider "cap" of puff (part exceeding $I = \alpha$)
- Size threshold governed by competition between diffusion of turbulence out of cap and total nonlinear growth in cap
- Sets scale $\sqrt{D/\gamma}$. Can derive $L_{min} \sim (I_0 - \alpha)^{-1/2}$



Avalanche threshold: analytical vs. simulation

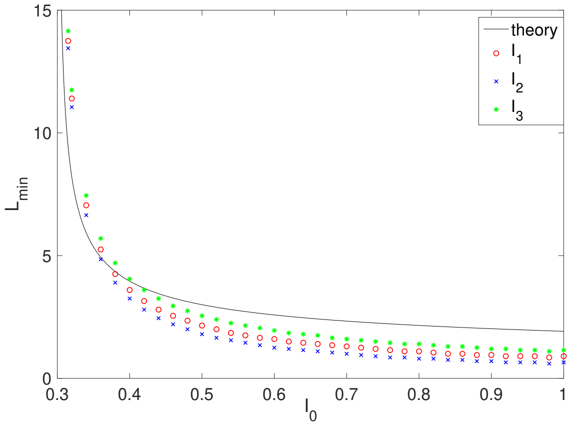


Figure Numerical result for threshold at $\alpha = 0.3$ for three types of initial condition (Gaussian (I_1), Lorentzian (I_2), parabola (I_3)), compared with analytical estimate

Bistable model: conclusions

- Bistable model rectifies issues with Fisher, is supported by evidence for subcritical turbulence
- Provides simple framework for understanding avalanching: local, intermittent exceedance of nonlinear instability by turbulent puffs. Threshold weak near marginal $\rightarrow$ triggered by noise?
- Key testable predictions: ballistic spreading into weakly linearly damped regions, power-law threshold for spreading of puffs

Note on experiments

- [Van Compernolle et al., 2015] created avalanches in experiment by locally perturbing plasma with source, measuring spatiotemporal response
- Suggest testing avalanche threshold in similar manner. How intense/large must source be?
- [Inagaki et al., 2013]: purported hysteresis between fluctuation intensity and driving gradient (no TB present)
- But if bistable, why does intensity relax after source turned off?
- Suggest more experiments à la Inagaki to investigate bistability

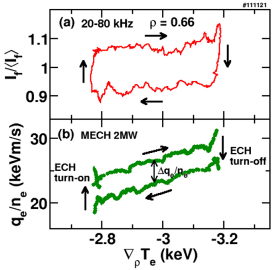
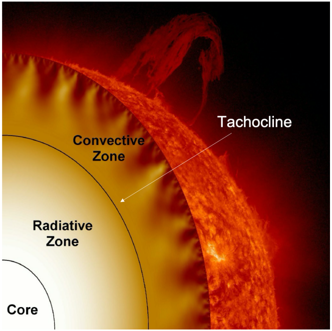


Figure Hysteresis between intensity and gradient, flux and gradient

Beta-plane MHD project

Solar tachocline

- Thin, radially-sheared layer at base of convection zone. Strongly turbulent
- Believed to be strongly involved in the solar dynamo
- Home to Ω-effect: shear drags poloidal field lines originating from core, converts to strong toroidal field
- Momentum transport crucial to problem of why tachocline exists. Friction or anti-friction? [Spiegel and Zahn, 1992, Gough and McIntyre, 1998]



β-plane MHD model

- Strong stratification in tachocline $\implies$ quasi-2D
- 2D magnetized incompressible turbulence in presence of planetary vorticity (Coriolis force) gradient: $2\boldsymbol{\Omega} = (0, 0, f + \beta y)$

$$\boxed{\begin{aligned} \partial_t \nabla^2 \psi + \beta \partial_x \psi &= \{\psi, \nabla^2 \psi\} - \{A, \nabla^2 A\} + \nu \nabla^4 \phi + \tilde{f} \\ \partial_t A &= \{\psi, A\} + \eta \nabla^2 A \end{aligned}}$$

- $\mathbf{v} = (\partial_y \psi, -\partial_x \psi, 0)$, $\mathbf{B} = (\partial_y A, -\partial_x A, 0)$
- $\{a, b\} = \partial_x a \partial_y b - \partial_y a \partial_x b$
- Note similarity to HW: β plays the role of $\partial_x \langle n \rangle$

Effect of (weak) mean field

- Tobias *et al.* (2007) assessed impact of weak mean field $b_0\hat{x}$ on zonal flow formation
- Above a critical b_0, turbulence is "Alfvénized." Reynolds-Maxwell stress $\langle\partial_x\psi\partial_y\psi\rangle - \langle\partial_x A\partial_y A\rangle \sim \sum_{\mathbf{k}}(|\mathbf{v}_{\mathbf{k}}|^2 - |\mathbf{B}_{\mathbf{k}}|^2)$ small $\implies$ no ZF
- η large enough $\implies$ quenches magnetic turbulence $\implies$ critical b_0 can be quite large

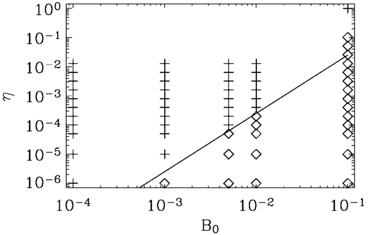


FIG. 5.—Scaling law for the transition between forward cascades (*diamonds*) and inverse cascades (*plus signs*). The line is given by B_0^2/η = constant.

Cross-helicity

- Previous analytical studies have neglected the effect of cross-helicity $\langle \mathbf{v} \cdot \mathbf{B} \rangle = -\langle A \nabla^2 \psi \rangle$. Often frozen at zero for simplicity, invoking usual conservation law
- However, Coriolis term explicitly breaks conservation:

$$\partial_t \langle A \nabla^2 \psi \rangle = -\beta \langle v_y A \rangle + \mathrm{dissipation}$$

- In this work: seek to elucidate the role of cross-helicity in this system. What is role in momentum transport?

Stationary value

As a start, can obtain stationary CH value from a simple calculation à la Zeldovich. Neglecting forcing:

$$\frac{1}{2}\partial_t\langle A^2\rangle = b_0\langle A\partial_x\psi\rangle - \eta\langle(\nabla A)^2\rangle$$

$$\implies \langle A\partial_x\psi\rangle_\infty = \frac{\eta}{b_0}\langle\tilde{b}^2\rangle$$

$$\partial_t\langle A\nabla^2\psi\rangle = -\beta\langle A\partial_x\psi\rangle + (\eta+\nu)\langle\nabla^2\psi\nabla^2 A\rangle$$

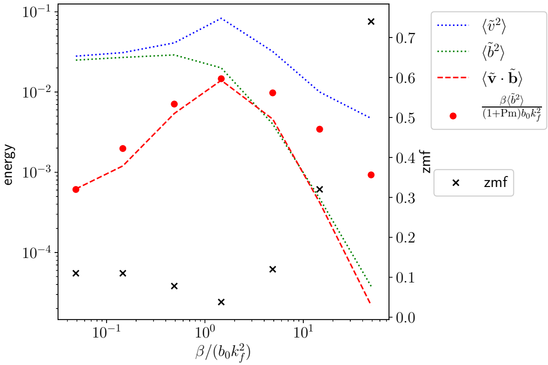
$$\implies \boxed{\langle A\nabla^2\psi\rangle_\infty \simeq \frac{\beta\langle\tilde{b}^2\rangle\ell_b\ell_v}{b_0(1+\mathrm{Pm})}}$$

where $\mathrm{Pm} \equiv \frac{\nu}{\eta}$

Note appearance of "magnetic Rhines" scale $k_{MR} = \sqrt{\frac{\beta}{b_0}}$, defines crossover of Rossby and Alfvén frequencies

Simulation results

- Simulate β-plane system with fixed $b_0 = 2$, $\eta = \nu = 10^{-4}$, $\epsilon = 0.01$, $k_f = 32$ at various β
- Transition to Rossby turb. begins around $k_{MR} = k_f$ $(\beta = b_0 k_f^2)$
- Good agreement with Zeldovich with $\ell = \ell_f$ (breaks down for large β as $\ell_b < \ell_f$)
- Transition presaged by increasing mean CH — suggests CH plays a role?



Weak turbulence theory

- Need spectra to determine transport. Seek closure of spectral equations that treats cross-helicity on equal footing with energy spectra
- Simplest approach: weak turbulence theory [Sagdeev and Galeev, 1969]. Treat nonlinear terms as triplet interactions between resonant linear modes
- Downside: dubious for small k_x or weak field
- Two eigenmodes in this system (Rossby-Alfvén)

$$\omega_{\pm} = \frac{\omega_\beta \pm \sqrt{4\omega_A^2 + \omega_\beta^2}}{2}$$

with $\omega_\beta = -\beta k_x / k^2$, $\omega_A = k_x b_0$

Spectra I

- Can write down spectral equations for correlators $C_{\mathbf{k}}^{\alpha\alpha'}$,* but very complicated. Hard to make progress
- Perturbation theory for small β doesn't work. β changes topology of resonant surfaces
- However, Rossby-Alfvén cross-correlator naturally oscillates at $\omega_+ - \omega_- = \Omega = \sqrt{4\omega_A^2 + \omega_\beta^2} \to$ time average is zero!
- We have

$$k^2 \,\mathrm{Re}(C_{\mathbf{k}}^{+-} e^{-i\Omega t}) = -\frac{1}{\Omega^2}\left(\omega_A^2(|\tilde{v}_{\mathbf{k}}|^2 - |\tilde{b}_{\mathbf{k}}|^2) + \omega_\beta \omega_A \,\mathrm{Re}\langle \tilde{\mathbf{v}}_{\mathbf{k}} \cdot \tilde{\mathbf{b}}_{-\mathbf{k}}\rangle\right)$$

$$\Longrightarrow \boxed{|\tilde{v}_{\mathbf{k}}|^2 - |\tilde{b}_{\mathbf{k}}|^2 = \frac{\beta}{b_0 k^2} \mathrm{Re}\langle \tilde{\mathbf{v}}_{\mathbf{k}} \cdot \tilde{\mathbf{b}}_{-\mathbf{k}}\rangle}$$

Time-averaged, stationary cross-helicity spectrum entirely determines momentum transport!

*$\langle \phi_{\mathbf{k}}^{\alpha} \phi_{\mathbf{k}'}^{\alpha'} \rangle = C_{\mathbf{k}}^{\alpha\alpha'} \delta(\mathbf{k} + \mathbf{k}') e^{-i(\omega_{\mathbf{k}}^{\alpha} - \omega_{\mathbf{k}}^{\alpha'})t}$

Spectra II

- Buildup of cross-helicity during transition thus linked to breakdown of Alfvénization condition $|\tilde{v}_{\mathbf{k}}|^2 = |\tilde{b}_{\mathbf{k}}|^2$
- Equivalently:

$$\frac{\langle \partial_t \tilde{v} \rangle_{\mathbf{k}}}{\langle \partial_t \tilde{b} \rangle_{\mathbf{k}}} = \frac{k_{\mathrm{MR}}^2}{k^2}.$$

$\implies$ Fluctuations kinetic for $\ell > \ell_{MR}$, magnetic for $\ell < \ell_{MR}$ [Diamond et al., 2007]

- Also have estimate (for $\beta \lesssim b_0 k_f^2$):

$$\langle \tilde{v}^2 \rangle - \langle \tilde{b}^2 \rangle \simeq \frac{\beta^2}{b_0^2 k_f^4} \frac{\langle \tilde{b}^2 \rangle}{1 + \mathrm{Pm}}$$

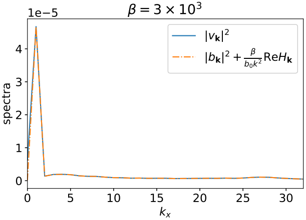


Figure Time-averaged, k_y-averaged spectra from simulation, confirming calculation. Note that spectra don't agree at $k_x = 0$ because $\Omega \to 0$

Conclusion

- Cross helicity is non-conserved in β-plane MHD. In presence of mean magnetic field, attains a finite stationary value
- In weak turbulence theory, stationary cross-helicity spectrum equivalent to Maxwell-Reynolds stress $\rightarrow$ determines momentum transport
- Have confirmed both of these calculations in simulation
- $H = \frac{\beta\langle\tilde{b}^2\rangle\ell_b\ell_v}{b_0(1+\mathrm{Pm})}$ could be very large for weak b_0, large Rm. Should study this case numerically! Flux of magnetic potential?
- CH spectrum related to turbulent emf, but need 3D to study dynamo

Final remarks: where does ML fit in with the other projects?

- Had hoped to use machine learning approach to study interactions between spreading and ZF (spreading breaks up ZF, ZF limits spreading?).
- But: diffusive mean field model $\langle \tilde{v}_x(\tilde{n} - \nabla^2\tilde{\phi})^2 \rangle = f(\varepsilon, \partial_x\varepsilon, \dots)$ didn't work. Spreading not important in adiabatic HW? Spreading not described by local model?
- Given similarities between beta-plane MHD and HW, might consider applying ML
- Issues: no adiabaticity, need to specify forcing, 1D model only makes sense when k_{MR} is large
- Final outlook: would like to apply ML methodology to other systems. 2D HW with generic α, 3D HW lowest-hanging fruits. Other systems with special spatial DOF?

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- Thank you to my committee
- Thank you to my lovely wife, Monica
- Thank you to my wonderful parents, Olle Heinonen and Leslie McDonald
- Countless other family, friends, mentors

Extra slides

Sketch of Hasegawa-Wakatani derivation

- Assume cold ions $T_i = 0$, use ion/electron continuity + $E \times B$ and ion polarization drifts + Ohm's law for parallel electron current + quasineutrality
-

$$\partial_t n_\alpha + \mathbf{v}_\alpha \cdot \nabla n_\alpha = 0$$

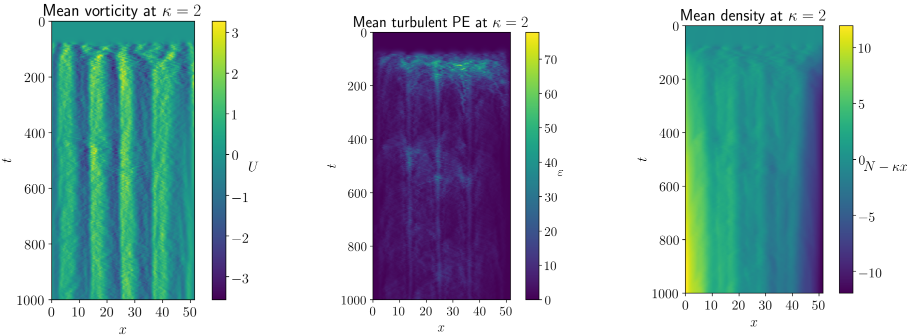
$$\text{force balance: } \mathbf{v}_i = \underbrace{-\frac{c}{B}\nabla\phi \times \hat{z}}_{\mathbf{v}_E} - \underbrace{\frac{c}{\omega_{ci} B}\frac{d\nabla\phi}{dt}}_{\text{polarization}} + \underbrace{\frac{\mu c}{\omega_{ci} B}\nabla^2(\nabla\phi)}_{\text{viscosity}}$$

$$\mathbf{v}_e = \mathbf{v}_E + v_{e,\|} \text{ (ignore pol. drift due to mass ratio)}$$

$$\eta J_{e,\|} = -\nabla_\| \phi + \frac{1}{e n_e}\nabla_\| p_e, \; p_e = n_e T_e \rightarrow \text{solve for } J_\|$$

- Sub above into continuity, use quasineutrality $n_e \simeq n_i$ ($\lambda_D \ll \ell$)

Compare to zonally averaged 2D DNS



1D resembles simplified version of DNS. One key difference: 3-field model equivalent to taking stationary "best-fit" spectrum. Some system memory lost

Reynolds stress: intensity scaling

- Whereas learned Γ is essentially $\propto \varepsilon$, Π scaling with ε is nontrivial
- Learned exponent is 1 for small intensity, close to zero for large intensity
- Jibes with intuition from strong turbulence theory

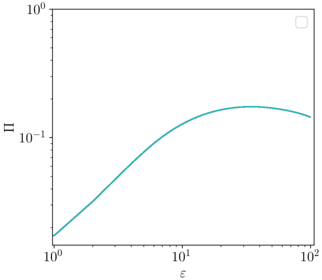


Figure Reynolds stress dependence on gradients at fixed ε, U, U''

Drift-wave/zonal flow system

- Drift-wave turbulence features complex interaction between mean density profile, ZF, and turbulence
- Dynamics controlled by cross-correlations between fluctuating quantities (turbulent fluxes)
- Difficult to calculate, requires successive, often dubious approximations to make progress

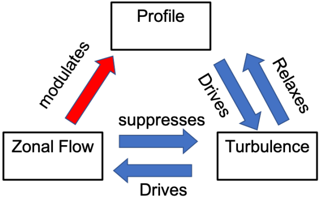


Figure Feedback loop illustrating interaction of mean fields in DW turbulence

Reynolds stress: hyperviscosity

Hyperviscous term, crucial for stability, has small coefficient. Sensitive test of method

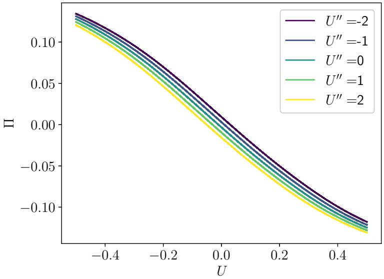


Figure U'' level curves of Reynolds stress as function of U, at fixed ε, U', N'

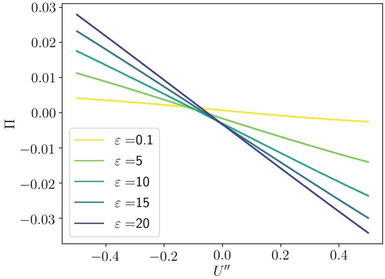


Figure ε level curves of Reynolds stress as function of U'', at fixed U, U', N''

Cousin models

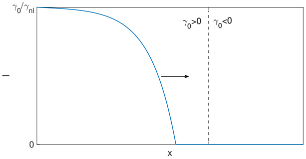
- Compare to bistable models for subcritical transition to fluid turbulence [Barkley et al., 2015, Pomeau, 2015].
- Compare to [Gil and Sornette, 1996] model for sandpile avalanches

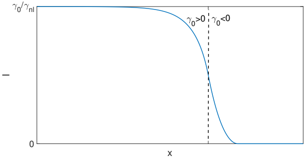
$$\partial_t S = \gamma \left(|\partial_x h| / g_c - 1 \right) S + \beta S^2 - S^3 + \partial_x (D_S S \partial_x S)$$
$$\partial_t h = \partial_x (D_h S \partial_x h).$$

- $S \leftrightarrow I$, $h \leftrightarrow p$
- Weak gradient coupling limit $D_p \ll D_I \Rightarrow$ our model
- Strong gradient coupling limit: I slaved to p. $\partial_x p \propto I^{-1} \Rightarrow$ linear term is $c - \gamma I$, where c is a constant which depends on BCs. Bistable again!

Penetration into stable zone for Fisher

- Consider spreading of turbulence from lin. unstable to lin. stable zone
- Simple model: $\gamma_1 = \gamma_g > 0$ for $x < 0$, $\gamma_1 = -\gamma_d < 0$ for $x > 0$
- Allow turbulent front to form in lefthand region and propagate
- In Fisher model, penetration is *weak*: forms stationary, exponentially-decaying profile with $\lambda \sim \sqrt{D_0/\gamma_{nl}} \sim \Delta_c$. Dubiously consistent with observation





Avalanche threshold (details)

- Strategy: assume initial puff is symmetric, has single max I_0 and single lengthscale L
- Expand intensity curve about max to quadratic order, plug into dynamical equation, integrate over extent of cap
- Result: growth if

$$L > L_{\min} = \sqrt{\frac{D(\alpha)I_0}{f(I_0) - \frac{1}{3}(I_0 - \alpha)f'(I_0)}} = \sqrt{\frac{3D\alpha I_0}{\gamma(I_0 - \alpha)((1 - 2\alpha)I_0 + \alpha)}}$$

- Power law $L_{min} \sim (I_0 - \alpha)^{-1/2}$

Evidence for subcriticality

- [Inagaki et al., 2013]: experiments demonstrate hysteresis between fluctuation intensity and driving gradient (no TB present). Suggests bistable S-curve relation?
- Turbulence subcritical in presence of strong perpendicular flow shear [Carreras et al., 1992, Barnes et al., 2011, van Wyk et al., 2016] or in the presence of magnetic shear [Biskamp and Walter, 1985, Drake et al., 1995]
- Profile corrugations [Waltz, 1985, Waltz, 2010] and phase space structures [Lesur and Diamond, 2013] can drive nonlinear instability

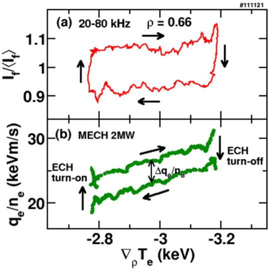


Figure Hysteresis between intensity and gradient, flux and gradient

Closure theory

- How to go from dynamical equations

$$\partial_t \phi_{\mathbf{k}}^{\alpha} + i\omega_{\mathbf{k}} \phi_{\mathbf{k}}^{\alpha} = \frac{1}{2} \sum_{\mathbf{k}'+\mathbf{k}''=\mathbf{k}} M_{\mathbf{k},\mathbf{k}',\mathbf{k}''}^{\alpha\beta\gamma} \phi_{\mathbf{k}'}^{\beta} \phi_{\mathbf{k}''}^{\gamma}$$

  to equations for spectra $\langle \phi_{\mathbf{k}}^{\alpha} \phi_{-\mathbf{k}}^{\alpha'} \rangle$?
- Multiplying thru by $\phi_{\mathbf{k}'}^{\alpha'}$ yields equation which involves third-order moments $\langle \phi\phi\phi \rangle$, third-order moment equation involves fourth-order moments, etc.
- "Closure problem": how to express higher-order moments in terms of lower-order moments and close system?
- DIA (Kraichnan): $\langle \phi_{\mathbf{k}} \phi_{\mathbf{k}'} \phi_{\mathbf{k}''} \rangle \simeq \langle \phi_{\mathbf{k}}^{(c)} \phi_{\mathbf{k}'} \phi_{\mathbf{k}''} \rangle + \ldots$ where $\phi_{\mathbf{k}}^{(c)}$ coherent to direct beat $\phi_{\mathbf{k}'} \phi_{\mathbf{k}''}$. Equiv. to 1-loop renormalization

Spectral equations

Weak turb. spectral equations for arbitrary number of scalar fields ϕ^α (in eigenbasis) can be derived straightforwardly:

$$\partial_t C_{\mathbf{k}}^{\alpha\alpha'} = \sum_{\mathbf{k}'+\mathbf{k}''=\mathbf{k}} \sum_{\beta\gamma} \Bigg[|M_{\mathbf{k},\mathbf{k}',\mathbf{k}''}^{\alpha\beta\gamma}|^2 C_{\mathbf{k}'}^{\beta\beta} C_{\mathbf{k}''}^{\gamma\gamma} \delta(\omega_{\mathbf{k}}^{\alpha} - \omega_{\mathbf{k}'}^{\beta} - \omega_{\mathbf{k}''}^{\gamma}) \delta_{\alpha\alpha'}$$

$$+ M_{\mathbf{k},\mathbf{k}',\mathbf{k}''}^{\alpha\beta\gamma} M_{\mathbf{k}',\mathbf{k},-\mathbf{k}''}^{\beta\alpha\gamma} C_{\mathbf{k}}^{\alpha\alpha'} C_{\mathbf{k}''}^{\gamma\gamma} \left(\pi\delta(\omega_{\mathbf{k}}^{\alpha} - \omega_{\mathbf{k}'}^{\beta} - \omega_{\mathbf{k}''}^{\gamma}) + i\mathcal{P} \frac{1}{\omega_{\mathbf{k}}^{\alpha} - \omega_{\mathbf{k}'}^{\beta} - \omega_{\mathbf{k}''}^{\gamma}} \right)$$

$$+ M_{\mathbf{k},\mathbf{k}',\mathbf{k}''}^{\alpha'\beta\gamma*} M_{\mathbf{k}',\mathbf{k},-\mathbf{k}''}^{\beta\alpha'\gamma*} C_{\mathbf{k}}^{\alpha\alpha'} C_{\mathbf{k}''}^{\gamma\gamma} \left(\pi\delta(\omega_{\mathbf{k}}^{\alpha'} - \omega_{\mathbf{k}'}^{\beta} - \omega_{\mathbf{k}''}^{\gamma}) - i\mathcal{P} \frac{1}{\omega_{\mathbf{k}}^{\alpha'} - \omega_{\mathbf{k}'}^{\beta} - \omega_{\mathbf{k}''}^{\gamma}} \right) \Bigg].$$

where $\langle \phi_{\mathbf{k}}^{\alpha} \phi_{\mathbf{k}'}^{\alpha'} \rangle = C^{\alpha\alpha'} \delta(\mathbf{k} + \mathbf{k}') e^{-i(\omega_{\mathbf{k}}^{\alpha} - \omega_{\mathbf{k}}^{\alpha'})t}$, $M_{\mathbf{k}\mathbf{k}'\mathbf{k}''}^{\alpha\beta\gamma}$ are symmetrized nonlinear coupling coefficients. PV integrals vanish in case of real coupling coefficients and a single field, recover Sagdeev-Galeev result.

MHD snapshots at $\beta = 3000$ at $t = 400$

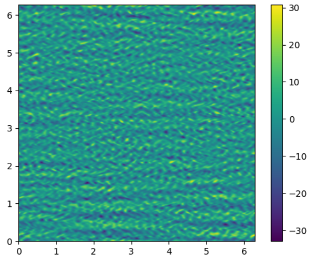


Figure $\nabla^2\psi$

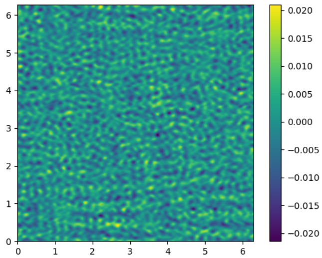


Figure A

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