Intrinsically Multi-Scale Microturbulence in a Stochastic Magnetic Field

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Asia-Pacific Transport Working Group M
US-EU Transport Task Force Wor

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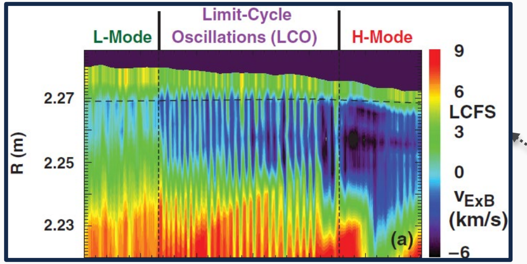
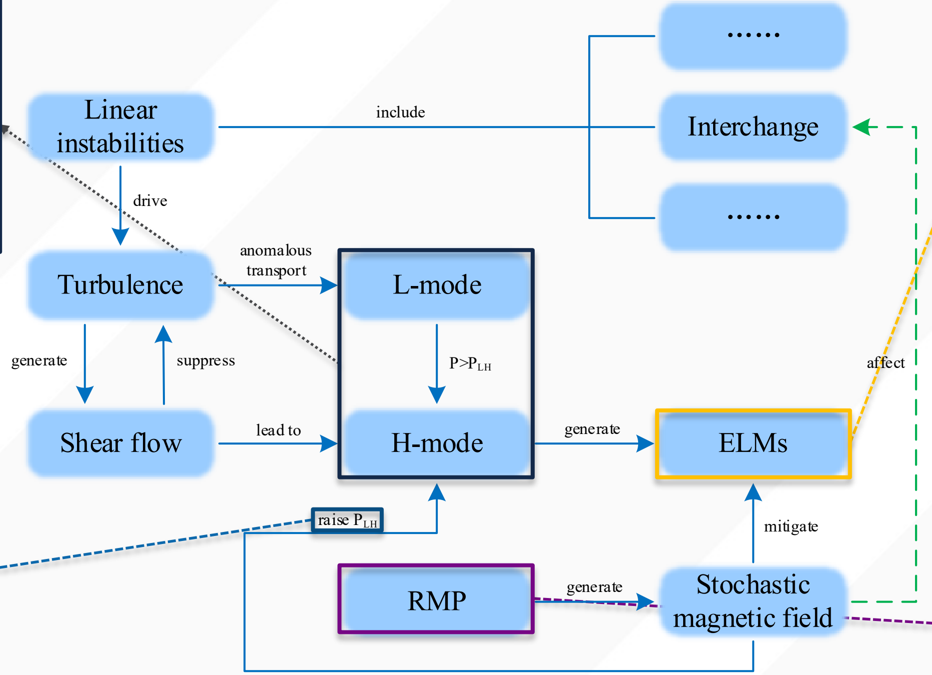
OUTLINE

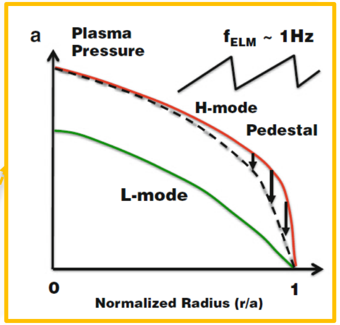
- Motivation: ELM, RMP & Stochastic Magnetic Field
- (Ancient) History: How is instability process modified?
- Formulation: A Simple Model Maintaining $\nabla \cdot \boldsymbol{J} = 0$
- Analysis: Physical Picture Behind the Calculation
- Conclusion : Where Things Stand & Lessons learned
- Future: What Next?

MOTIVATION: ELM, RMP & STOCHASTIC MAGNETIC FIELD

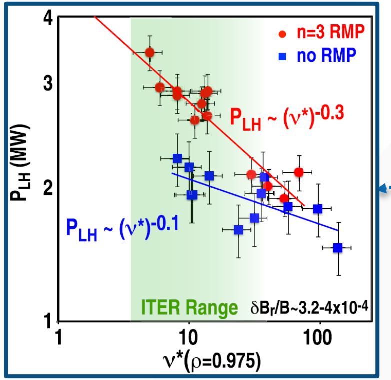


Time history of $\boldsymbol{E} \times \boldsymbol{B}$ velocity across the plasma edge in a L-H transition [1]



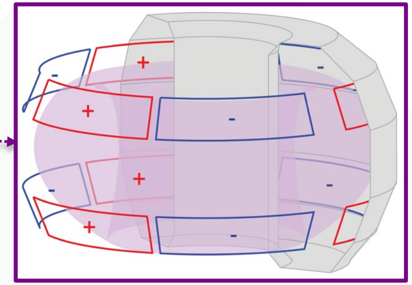


Schematics of L-mode and H-mode pressure profiles and the pressure collapse near the edge region due to ELM [3]



L-H transition power threshold P_{LH} versus collisionality v^* [2]

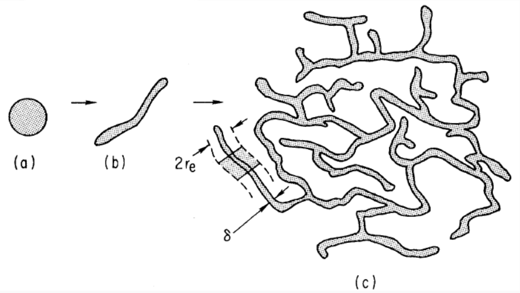
1. L. Schmitz et al, 2012. Physical review letters, 108(15), p.155002.
2. L. Schmitz et al, 2019. Nuclear Fusion, 59(12), p. 126010.
3. M. Kikuchi, and M. Azumi, 2015. Frontiers in fusion research II. Heidelberg etc.: Springer.



DIII-D I-coil system [2]

(ANCIENT) HISTORY: HOW IS INSTABILITY PROCESS MODIFIED?

- Background: People realized that stochastic magnetic field may affect certain phenomena in tokamaks in the late 1970s. For example, anomalous electron heat transport. [1]



The evolution of area mapping of field lines and guiding-center trajectories (a test particle picture)

→ test particle model

- Classic of Ancient History: Tearing modes in a braided magnetic field [2]. The main influence of stochastic magnetic field on macroscopic tearing modes lies in anomalous electron viscosity coefficient $\bar{\mu}$, i.e., the diffusion of current

$$E_{\|} = \eta J_{\|} - \bar{\mu}(m/ne^2)\nabla_{\perp}^2 J_{\|}.$$

Defects: 1. Physics of μ? How derive? 2. Lack of (or too simple) micro-macro connection

1. A. B. Rechester, and M. N. Rosenbluth, 2020. In *Hamiltonian Dynamical Systems* (pp. 684-687). CRC Press.
2. P. K. Kaw, E. J. Valeo, and P. H. Rutherford, 1979. *Physical Review Letters*, *43*(19), p.1398.

FORMULATION: A SIMPLE MODEL MAINTAINING $\nabla \cdot \boldsymbol{J} = 0$

Target: Construct a simple model to get insights and guide simulations

The model should
- maintain $\nabla \cdot \boldsymbol{J} = 0$
- connect micro and macro scales
- be tractable

→ **Electrostatic interchange**

- Linearized vorticity equation

$$\underbrace{-(\rho_0/B_0^2)\partial_t \nabla_\perp^2 \phi}_{\nabla_\perp \cdot \boldsymbol{J_{pol}}} \underbrace{-(g/B_0)\partial_y p}_{\nabla_\perp \cdot \boldsymbol{J_{PS}}} + \underbrace{\boldsymbol{b_0} \cdot \nabla J_\parallel}_{\nabla_\parallel J_\parallel} = 0$$

→ $\nabla \cdot \boldsymbol{J} = 0$

- Electrostatic Ohm's law of resistive MHD

$$E_\parallel = -\nabla_\parallel \phi = \eta_\parallel J_\parallel$$

- Linearized pressure equation

$$\partial_t p - (\nabla \phi \times \hat{\boldsymbol{z}})/B_0 \cdot \nabla p_0 = 0$$

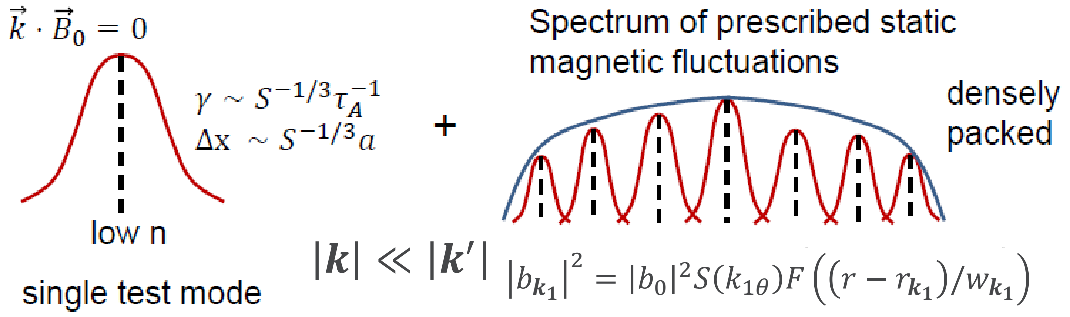
FORMULATION: A SIMPLE MODEL MAINTAINING $\nabla \cdot \boldsymbol{J} = 0$

Introduce a static stochastic magnetic field $\tilde{\boldsymbol{b}} = \tilde{\boldsymbol{B}}_\perp / B_0 = \sum_{m,n} \tilde{\boldsymbol{b}}_{m,n}(x') e^{i(m\theta - n\phi)}$

Total field = Main field + Randomly tilted lines, i.e.

$$\nabla_\parallel = \nabla_\parallel^{(0)} + \tilde{\boldsymbol{b}} \cdot \nabla_\perp \qquad (\nabla \cdot \tilde{\boldsymbol{b}} = 0)$$

Model: a low-k single test mode + a high-k stochastic magnetic field background



$$\frac{\partial}{\partial t} \nabla_\perp^2 \bar{\varphi} = -\frac{S}{\tau_A} \left(\nabla_\parallel^{(0)} + \tilde{\boldsymbol{b}} \cdot \nabla_\perp \right)^2 \bar{\varphi} - \frac{g B_0}{\rho_0} \frac{\partial \bar{p}}{\partial y}$$

FORMULATION: A SIMPLE MODEL MAINTAINING $\nabla \cdot \boldsymbol{J} = 0$

Remember: we want to keep $\nabla \cdot \boldsymbol{J} = 0$ at all scales.

If only $\widetilde{\boldsymbol{b}}$ and $\bar{\varphi}$, $\nabla \cdot \boldsymbol{J} = 0$ is not guaranteed!

At micro scale:

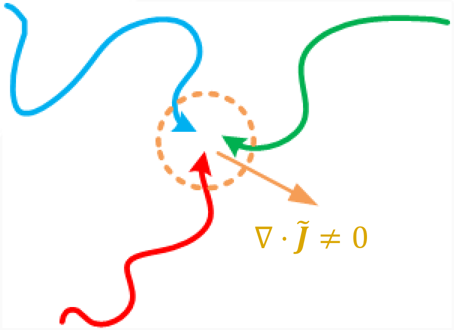
$$\tilde{\boldsymbol{J}} = \tilde{\boldsymbol{J}}_{\parallel} = \tilde{\boldsymbol{J}}_{\parallel^0} + \tilde{\boldsymbol{J}}_{\perp} = -\frac{1}{\eta_{\parallel}}(\widetilde{\boldsymbol{b}} \cdot \nabla_{\perp})\bar{\phi}\boldsymbol{b_0} - \frac{1}{\eta_{\parallel}}\nabla_{\parallel}^{(0)}\bar{\phi}\widetilde{\boldsymbol{b}}$$

$$\nabla \cdot \tilde{\boldsymbol{J}} = \nabla_{\parallel}^{(0)}\tilde{J}_{\parallel^0} + \nabla_{\perp} \cdot \tilde{\boldsymbol{J}}_{\perp} = -\frac{1}{\eta_{\parallel}}\left\{\nabla_{\parallel}^{(0)}\left[(\widetilde{\boldsymbol{b}} \cdot \nabla_{\perp})\bar{\phi}\right] + (\widetilde{\boldsymbol{b}} \cdot \nabla_{\perp})\nabla_{\parallel}^{(0)}\bar{\phi}\right\} \neq 0$$

$\nabla \cdot \boldsymbol{J} = 0$ no longer holds!

There must be something we've missed at small scales.

What is the missing piece?



FORMULATION: A SIMPLE MODEL MAINTAINING $\nabla \cdot \boldsymbol{J} = 0$

Insights from the Classic: Kadomtsev and Pogutse '78 [1]:

Electron heat flux is divergence free at all scales → $\nabla \cdot \boldsymbol{q} = 0$

where $\boldsymbol{q} = -\chi_{\parallel}\nabla_{\parallel}T\hat{\boldsymbol{b}} - \chi_{\perp}\nabla_{\perp}T$ $\quad \hat{\boldsymbol{b}} = \hat{\boldsymbol{b}}_{\mathbf{0}} + \tilde{\boldsymbol{b}}$ $\quad \nabla_{\parallel} = \nabla_{\parallel}^{(0)} + \tilde{\boldsymbol{b}} \cdot \nabla_{\perp}$

The idea is, to maintain $\nabla \cdot \boldsymbol{q} = 0$, there must be a temperature fluctuation $\tilde{T}$.

Analogy	K&P	C&D
Base state	$\langle T(r) \rangle$	$\bar{\varphi}_{\boldsymbol{k}}$
External fluctuation	$\tilde{\boldsymbol{b}}$	$\tilde{\boldsymbol{b}}$
Constraint	$\nabla \cdot \boldsymbol{q} = 0$	$\nabla \cdot \boldsymbol{J} = 0$
Resulting fluctuation	$\tilde{T}$	$???$

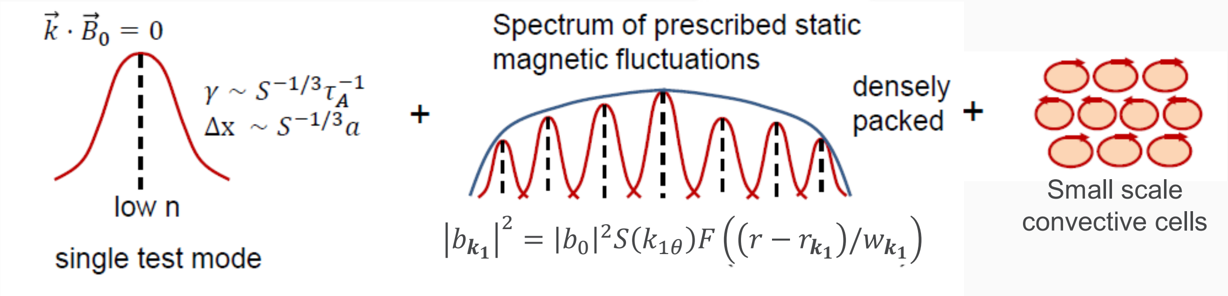
Hint: in this story, $\tilde{\boldsymbol{b}}$ could induce an electrostatic potential fluctuation $\tilde{\varphi}$

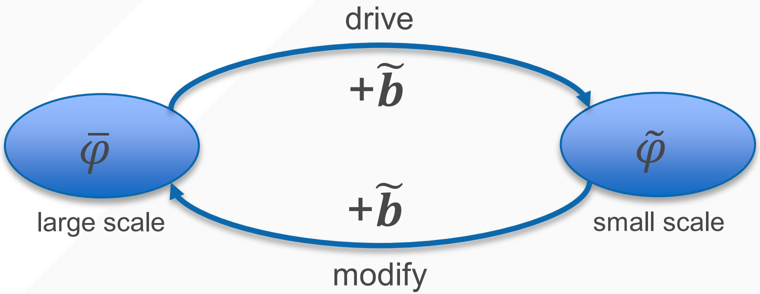
Therefore, to maintain $\nabla \cdot \boldsymbol{J} = 0$, electrostatic convective cells should be considered.[2]

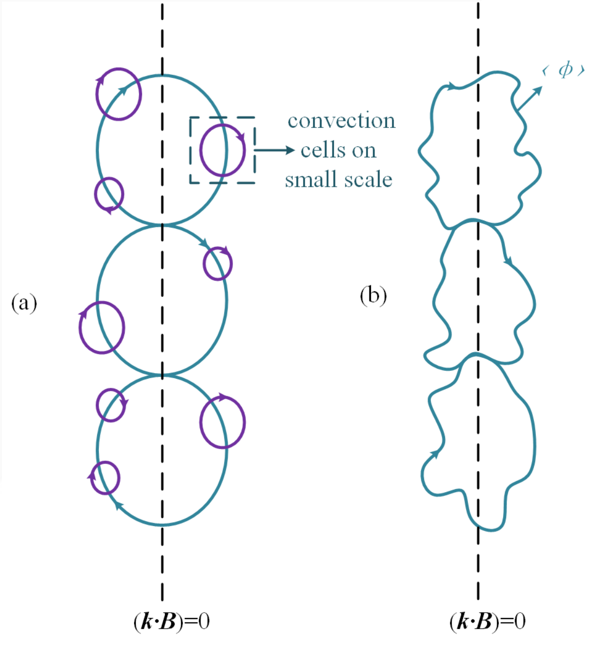
1. B. B. Kadomtsev, and O. P. Pogutse, 1979. *Plasma Physics and Controlled Nuclear Fusion Research 1978, Volume 1*, *1*, pp.649-662.
2. P. Beyer, X. Garbet, and P. Ghendrih, 1998. Physics of Plasmas, 5(12), pp.4271-4279.

FORMULATION: A SIMPLE MODEL MAINTAINING $\nabla \cdot J = 0$

The actual model is







Long wavelength cell in presence of short wavelength cells and $\tilde{b}$

Further, these small-scale convective cells drive a turbulent viscosity ν and a turbulent diffusivity χ.

FORMULATION: A SIMPLE MODEL MAINTAINING $\nabla \cdot J = 0$

By using **method of averaging**, the model is described by

$$①\left(\frac{\partial}{\partial t} - \nu\nabla_\perp^2\right)\nabla_\perp^2\bar{\varphi} = -\frac{S}{\tau_A}\left[\nabla_\parallel^{(0)^2}\bar{\varphi} + \underbrace{(\nabla_\perp \cdot \langle\tilde{\boldsymbol{b}}\tilde{\boldsymbol{b}}\rangle)\cdot\nabla_\perp\bar{\varphi}}_{(1)} + \underbrace{\left\langle\nabla_\parallel^{(0)}\nabla_\perp\cdot(\tilde{\boldsymbol{b}}\tilde{\varphi})\right\rangle}_{(2)} + \underbrace{\left\langle\nabla_\perp\cdot\left(\tilde{\boldsymbol{b}}\nabla_\parallel^{(0)}\tilde{\varphi}\right)\right\rangle}_{(3)}\right] - \frac{gB_0}{\rho_0}\frac{\partial\bar{p}}{\partial y}$$

$$②\left(\frac{\partial}{\partial t} - \nu\nabla_\perp^2\right)\nabla_\perp^2\tilde{\varphi} + \frac{S}{\tau_A}\nabla_\parallel^{(0)^2}\tilde{\varphi} + \frac{gB_0}{\rho_0}\frac{\partial\tilde{p}}{\partial y} = -\frac{S}{\tau_A}\left[(\tilde{\boldsymbol{b}}\cdot\nabla_\perp)\nabla_\parallel^{(0)}\bar{\varphi} + \nabla_\parallel^{(0)}(\tilde{\boldsymbol{b}}\cdot\nabla_\perp)\bar{\varphi}\right]$$

$$③\left(\frac{\partial}{\partial t} - \chi\nabla_\perp^2\right)\tilde{p} - \frac{\nabla\tilde{\varphi}\times\hat{\boldsymbol{z}}}{B_0}\cdot\nabla p_0 = 0 \quad ④\left(\frac{\partial}{\partial t} - \chi\nabla_\perp^2\right)\hat{p} - \frac{\nabla\bar{\varphi}\times\hat{\boldsymbol{z}}}{B_0}\cdot\nabla p_0 = 0 \quad \langle A\rangle = \bar{A} = \frac{1}{(2\pi)^2}\iint d\theta d\phi e^{-i(m\theta-n\phi)}A$$

Some assumptions/observations:

- $\bar{\varphi}$: low $\boldsymbol{k}$, slow interchange approximation $(1/w_{\boldsymbol{k}}^2 \gg k_y^2)$
- $\tilde{\varphi}$: high $\boldsymbol{k}'$, fast interchange approximation $(1/w_{\boldsymbol{k}'}^2 \ll k_y'^2)$

$\rightarrow$ $k_y^2 \ll \frac{1}{w_{\boldsymbol{k}}^2} \ll \frac{1}{w_{\boldsymbol{k}'}^2} \ll k_y'^2$

- $\tilde{\varphi}$ is driven by the beat of $\tilde{\boldsymbol{b}}$ and $\bar{\phi}$, thus macro scale and micro scale are now connected.
- As $\tilde{\boldsymbol{b}}$ is stationary, $\tilde{\varphi}$ is saturated by ν and χ, which originate from micro convective cells.

ANALYSIS: PHYSICAL PICTURE BEHIND THE CALCULATION

Main equation:

$$① \left(\frac{\partial}{\partial t} - \nu\nabla_\perp^2\right)\nabla_\perp^2\bar{\varphi} + \frac{S}{\tau_A}\underbrace{\left(\nabla_\perp\cdot\langle\tilde{\boldsymbol{b}}\tilde{\boldsymbol{b}}\rangle\right)\cdot\nabla_\perp\bar{\varphi}}_{(1)} = -\frac{S}{\tau_A}\left[\nabla_\parallel^{(0)^2}\bar{\varphi} + \underbrace{\left\langle\nabla_\parallel^{(0)}\tilde{\boldsymbol{b}}\cdot\nabla_\perp\tilde{\varphi}\right\rangle}_{(2)} + \underbrace{\left\langle\left(\tilde{\boldsymbol{b}}\cdot\nabla_\perp\right)\nabla_\parallel^{(0)}\tilde{\varphi}\right\rangle}_{(3)}\right] - \frac{gB_0}{\rho_0}\frac{\partial\bar{p}_1}{\partial y}$$

To determine $\langle\tilde{b}\tilde{\varphi}\rangle$, we need find linear response of $\tilde{\varphi}$ to $\tilde{\boldsymbol{b}}$. Another equation!

The vorticity equation at micro scale resembles a quantum harmonics under a weak drive.

$$-2\nu k_{2\theta}^2\frac{\partial^2}{\partial x_2^2}\tilde{\varphi}_{\boldsymbol{k}_2}(x_2) + \frac{S}{\tau_A}\frac{k_{2\theta}^2 x_2^2}{L_s^2}\tilde{\varphi}_{\boldsymbol{k}_2}(x_2) - \left(\frac{gp_0}{\chi\rho_0 L_p} - \nu k_{2\theta}^4\right)\tilde{\varphi}_{\boldsymbol{k}_2}(x_2)$$
$$= i\frac{S}{\tau_A}\left[(\partial_x k_\parallel)\bar{\varphi}_{\boldsymbol{k}}(x) + (k_{2\parallel} + k_\parallel)\partial_x\bar{\varphi}_{\boldsymbol{k}}(x)\right]\tilde{b}_{r\,(\boldsymbol{k}_2-\boldsymbol{k})}(x_1)$$

By exploiting fast-interchange approximation and using quasi-linear theory, we obtain

$$\tilde{\varphi}_{\boldsymbol{k}_2}(x_2) = i\frac{S}{\tau_A}\int G(x_2|x_2')\left[(\partial_{x'}k_\parallel)\bar{\varphi}_{\boldsymbol{k}}(x') + (k_{2\parallel} + k_\parallel)\partial_{x'}\bar{\varphi}_{\boldsymbol{k}}(x')\right]\tilde{b}_{r\,(\boldsymbol{k}_2-\boldsymbol{k})}(x_1')dx_2',$$

where

$$G(x_2, x_2') = \sum_n \frac{\psi_{\boldsymbol{k}_2}^n(x_2)\psi_{\boldsymbol{k}_2}^n(x_2')}{\Lambda_{\boldsymbol{k}_2}^n - \Lambda_{\boldsymbol{k}_2}}.$$

→ Eigen function of QHO

ANALYSIS: PHYSICAL PICTURE BEHIND THE CALCULATION

Third order magnetic torques:

(1): $\frac{S}{\tau_A}\left(\nabla_\perp \cdot \langle \tilde{\boldsymbol{b}}\tilde{\boldsymbol{b}} \rangle\right) \cdot \nabla_\perp \bar{\varphi} = \frac{S}{\tau_A}\partial_x |b_r|^2 \partial_x \bar{\varphi}_{\boldsymbol{k}}(x)$ → magnetic vorticity damping

→ 3^{rd} order $\nabla_\parallel J_\parallel$

$\gamma_{\boldsymbol{k}}\partial_x^2 \bar{\varphi}_{\boldsymbol{k}} + \frac{S}{\tau_A}\partial_x |b_r|^2 \partial_x \bar{\varphi}_{\boldsymbol{k}}(x)$ → Enhance inertia

As $\frac{S}{\tau_A}\left|\frac{\tilde{B}_{rk'}}{B_0}\right|^2 \sim \frac{V_A^2}{\eta}\frac{k_\theta'^2}{L_S^2} W_I'^4$ → $\frac{S}{\tau_A}\partial_x\left(\left|\tilde{b}_r\right|^2 \partial_x \bar{\varphi}\right) \sim \frac{v_A^2}{\eta}\frac{k_y^2}{L_S^2}\frac{w_I'^4}{(\Delta x)^2}\bar{\varphi}$ (3^{rd} order magnetic torque)

$(\nabla_\parallel J_\parallel)^{(1)} \sim \frac{S}{\tau_A}{\nabla_\parallel^{(0)}}^2 \bar{\varphi} \sim \frac{v_A^2}{\eta}\frac{k_y^2}{L_S^2}(\Delta x)^2$

$w_I' \equiv$ island width for stochastic field
$\Delta x \equiv \bar{\varphi}$ layer width

When $w_I' \sim \left[\frac{k_y^2}{{k_y'}^2}(\Delta x)^4\right]^{\frac{1}{4}}$, 3rd order magnetic torque balances 1st order. This is a reminiscent of Rutherford '73.[1] The ratio $(k_y^2/{k_y'}^2)$ is due to the multi-scale character.

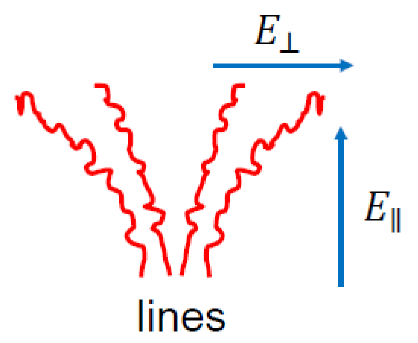
1. P. H. Rutherford, 1973. *The Physics of Fluids*, *16*(11), pp.1903-1908.

ANALYSIS: PHYSICAL PICTURE BEHIND THE CALCULATION

$$(2) = \left\langle \nabla_{\parallel}^{(0)} \nabla_{\perp} \cdot (\tilde{\boldsymbol{b}} \tilde{\varphi}) \right\rangle = \left\langle -\nabla_{\parallel}^{(0)} (\underline{\tilde{\boldsymbol{b}} \cdot \tilde{\boldsymbol{E}}_{\perp}}) \right\rangle$$

$$(3) = \left\langle \nabla_{\perp} \cdot \left(\tilde{\boldsymbol{b}} \nabla_{\parallel}^{(0)} \tilde{\varphi} \right) \right\rangle = -\nabla_{\perp} \cdot \underline{\langle \tilde{\boldsymbol{b}}_{\perp} \tilde{E}_{\parallel 0} \rangle}$$

E field projections along wandering tilting lines



- Perpendicular electric field $\tilde{\boldsymbol{E}}_{\perp}$ generates a parallel current.
- Parallel electric field $\tilde{E}_{\parallel 0}$ generates a perpendicular current

What is ν?

Recall equation $\hat{L}_{\boldsymbol{k}+k'} \tilde{\varphi}_{\boldsymbol{k}+k'} = C \tilde{b}_{\boldsymbol{k}'} \bar{\varphi}_{\boldsymbol{k}}$.
Assume $\tilde{\varphi}$ is near marginal in presence of weak excitation, then

$$\hat{L}_{\boldsymbol{k}+k'} \tilde{\varphi}_{\boldsymbol{k}+k'} \approx 0$$

Since $\tilde{\varphi}$ must be saturated by ν → $\nu = \left(g / L_p {k'_\theta}^4 \right)^{1/2}$

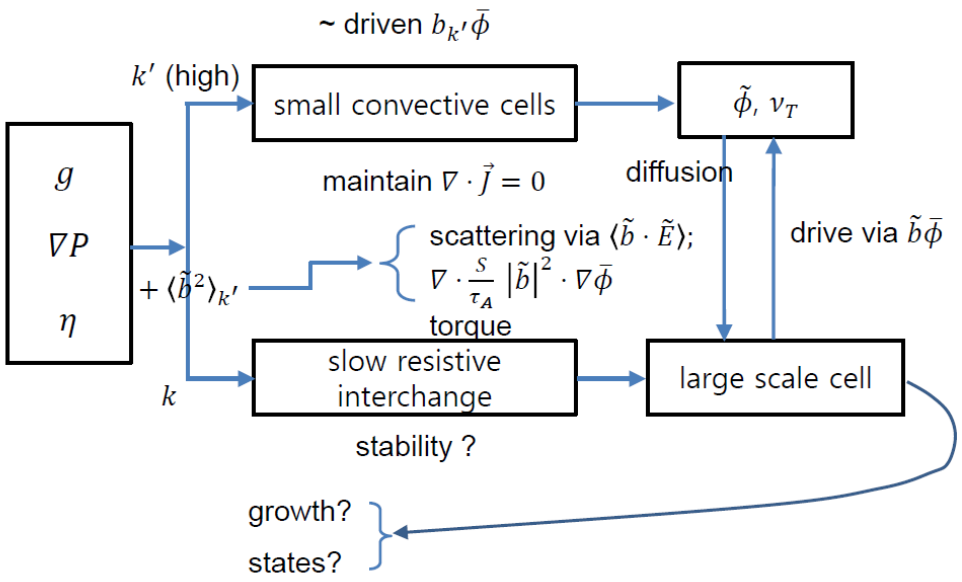
Above equation just provides a basic value of ν. The correction is given by the following closure

$$\nu = \sum_{\boldsymbol{k}_1} \left| \tilde{v}_{\boldsymbol{k}_1} \right|^2 \tau_{\boldsymbol{k}_1}$$

CONCLUSION : WHERE THINGS STAND & LESSONS LEARNED

- Integro-differential equation for $\bar{\varphi}$ evolution in presence of $|b_{k'}|^2$
- Effect and physics of the third-order magnetic torque are clear.
- Can formulate perturbation theory for $\gamma_k \to \gamma_k^{(0)} + \gamma_k^{(1)}$ (not finished yet)
- Obtain the value of ν

The Feedback Loop



$$\nu \approx \sum_{k_1} |c_{k_1}|^2 \langle \tilde{b}^2 \rangle_{k_1} |\bar{\varphi}|^2 \gamma_{k_1}^{-1} / \left[k_{1\theta}^2 - g k_{1\theta}^2 / \left(L_p \left(\nu k_{1\theta}^2 \right)^2 \right) \right]$$

CONCLUSION : WHERE THINGS STAND & LESSONS LEARNED

- Intrinsically a multi-scale problem:$\bar{\varphi}$; $\tilde{\varphi}$ and $\tilde{\boldsymbol{b}}$
- To maintain $\nabla \cdot \boldsymbol{J} = 0$ at all scales for prescribed $\tilde{\boldsymbol{b}}$ and instability$\bar{\varphi}$, $\tilde{\varphi}$(microscopic convective cells) is generated.
- This yields a non-trivial $\langle \tilde{b}\tilde{\varphi} \rangle$, i.e., electrostatic turbulence 'locks on' to magnetic perturbation.
- Identify magnetic vorticity damping effect (enhanced inertia)

$$inertia \rightarrow inertia + \frac{S}{\tau_A} \partial_x |b_r|^2 \partial_x \bar{\varphi}$$

- $w_I' \sim \left[(k_y^2 / {k_y'}^2)(\Delta x)^4 \right]^{1/4}$, when $(\nabla_\parallel J_\parallel)^{(1)} \sim (\nabla_\parallel J_\parallel)^{(3)}$.
  Magnetic vorticity damping is stronger than Rutherford's problem, for $k_y \ll k_y'$.

FUTURE: WHAT NEXT?

- Complete calculations of $\gamma_{\boldsymbol{k}}$ and ν to the first order by using perturbation theory.
- Determine the effects of $\left\langle \nabla_{\parallel}^{(0)} \nabla_{\perp} \cdot \left(\tilde{\boldsymbol{b}}\tilde{\varphi}\right)\right\rangle$ and $\left\langle \nabla_{\perp} \cdot \left(\tilde{\boldsymbol{b}}\nabla_{\parallel}^{(0)}\tilde{\varphi}\right)\right\rangle$. (a competition?)
- Another way to solve it? Schrodinger equation with 1-D random potential.
- Look at effects of stochastic magnetic field $\tilde{\boldsymbol{b}}$ on twisted slicing modes.

UC San Diego

Thank you

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