On the role of cross-helicity in β-plane MHD turbulence

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Supported by the U.S. Department of Energy under Award No. DE-FG02-04ER54738 and the European Research Council under grant No. 882340

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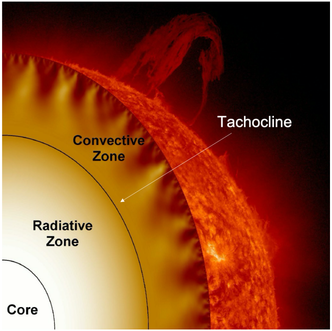
Preview

- Revisit analytic model for turbulence in solar tachocline
- More broadly is the simplest model for understanding interplay ZFs and magnetics, which is of great interest to the magnetic confinement problem
- Observe that cross-helicity is non-conserved. What can we learn by considering CH more carefully?
- Derive a simple estimate for the total stationary CH
- Sketch the weak turbulence theory for this (multi-field) system
- Using WT, derive a useful and interpretable constraint connecting CH to momentum transport

Solar tachocline

- Thin, radially-sheared layer at base of convection zone. Strongly turbulent
- Believed to be strongly involved in the solar dynamo
- Home to Ω-effect: shear drags poloidal field lines originating from core, converts to strong toroidal field
- Momentum transport crucial to problem of why tachocline exists. Friction or anti-friction? [Spiegel and Zahn, 1992, Gough and McIntyre, 1998]



β-plane MHD model

- Strong stratification in tachocline $\Longrightarrow$ quasi-2D
- 2D magnetized incompressible turbulence in presence of planetary vorticity (Coriolis force) gradient:
  $2\mathbf{\Omega} = (0, 0, f + \beta y)$

$$\boxed{\begin{aligned} \partial_t \nabla^2 \psi + \beta \partial_x \psi &= \{\psi, \nabla^2 \psi\} - \{A, \nabla^2 A\} + \nu \nabla^4 \phi + \tilde{f} \\ \partial_t A &= \{\psi, A\} + \eta \nabla^2 A \end{aligned}}$$

- $\mathbf{v} = (\partial_y \psi, -\partial_x \psi, 0)$, $\mathbf{B} = (\partial_y A, -\partial_x A, 0)$
- $\{a, b\} = \partial_x a \partial_y b - \partial_y a \partial_x b$
- Also serves as a toy model for drift-Alfvén turbulence

Effect of (weak) mean field

- Tobias *et al.* (2007) assessed impact of weak mean field $b_0\hat{x}$ on zonal flow formation
- Above a critical b_0, turbulence is "Alfvénized." Reynolds-Maxwell stress $\langle\partial_x\psi\partial_y\psi\rangle - \langle\partial_x A\partial_y A\rangle \sim \sum_{\mathbf{k}}(|\mathbf{v_k}|^2 - |\mathbf{B_k}|^2)$ small $\implies$ no ZF
- η large enough $\implies$ quenches magnetic turbulence $\implies$ critical b_0 can be quite large.

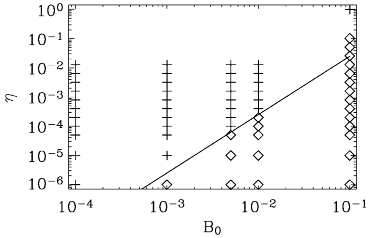


FIG. 5.—Scaling law for the transition between forward cascades (*diamonds*) and inverse cascades (*plus signs*). The line is given by B_0^2/η = constant.

Cross-helicity

- Previous analytical studies have neglected the effect of cross-helicity $\langle \mathbf{v} \cdot \mathbf{B} \rangle = -\langle A \nabla^2 \psi \rangle$. Often frozen at zero for simplicity, invoking usual conservation law
- However, Coriolis term explicitly breaks conservation:

$$\partial_t \langle A \nabla^2 \psi \rangle = -\beta \langle v_y A \rangle + \mathrm{dissipation}$$

- In this work: seek to elucidate the role of cross-helicity in this system. What is role in momentum transport?

Stationary value

As a start, can obtain stationary CH value from a simple calculation à la Zeldovich. Neglecting forcing:

$$\frac{1}{2}\partial_t \langle A^2 \rangle = b_0 \langle A \partial_x \psi \rangle - \eta \langle (\nabla A)^2 \rangle$$

$$\implies \langle A \partial_x \psi \rangle_\infty = \frac{\eta}{b_0} \langle \tilde{b}^2 \rangle$$

$$\partial_t \langle A \nabla^2 \psi \rangle = -\beta \langle A \partial_x \psi \rangle + (\eta + \nu) \langle \nabla^2 \psi \nabla^2 A \rangle$$

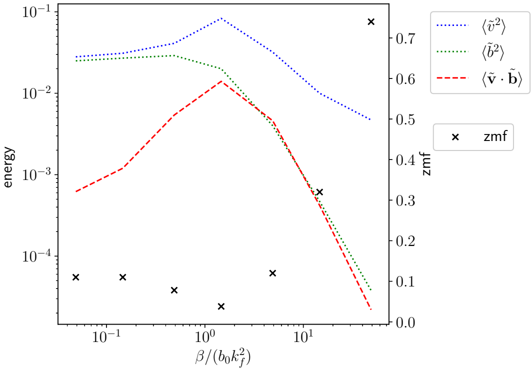
$$\implies \boxed{\langle A \nabla^2 \psi \rangle_\infty \simeq \frac{\beta \langle \tilde{b}^2 \rangle \ell_b \ell_v}{b_0 (1 + \mathrm{Pm})}}$$

where $\mathrm{Pm} \equiv \frac{\nu}{\eta}$

Note appearance of “magnetic Rhines” scale $k_{MR} = \sqrt{\frac{\beta}{b_0}}$, defines crossover of Rossby and Alfvén frequencies

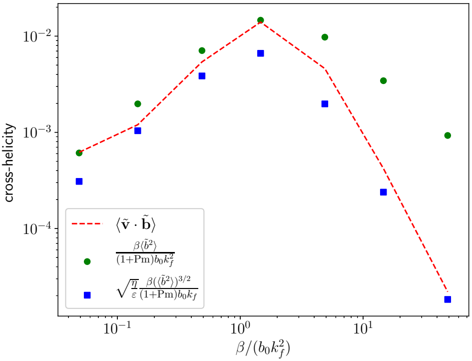
Simulation results

- Simulate β-plane system with fixed $b_0 = 2$, $\eta = \nu = 10^{-4}$, $\varepsilon = 0.01$, $k_f = 32$ at various β
- $\mathrm{Rm} \sim 6000 - 15000$
- Transition to Rossby turb. begins around $k_{MR} = k_f$ $(\beta = b_0 k_f^2)$
- Transition presaged by increasing mean CH — suggests CH plays a role?



Simulation results: comparison to Zel'dovich

- Taking $\ell_v = \ell_b = \ell_f$ in stationary CH estimate yields good agreement for $k_{MR} \lesssim k_f$
- At large β, $\ell_b \ll \ell_f$. There a better estimate is a magnetic Taylor microscale $\ell_b = \sqrt{\eta/\varepsilon\langle\tilde{b}^2\rangle}$.



Weak turbulence theory

- Need spectra to determine transport. Seek closure of spectral equations that treats cross-helicity on equal footing with energy spectra
- Simplest approach: weak turbulence theory [Sagdeev and Galeev, 1969]. Treat nonlinear terms as triplet interactions between resonant linear modes
- Downside: fails when linear frequency is small → can't describe $k_x \to 0$ limit or weak field
- Two eigenmodes in this system (Rossby-Alfvén)

$$\omega_\pm = \frac{\omega_\beta \pm \sqrt{4\omega_A^2 + \omega_\beta^2}}{2}$$

with $\omega_\beta = -\beta k_x / k^2$, $\omega_A = k_x b_0$

Spectral equations

- Generalization of weak turb. spectral equations for arbitrary number of scalar fields ϕ^α rarely seen, but can be derived:

$$\partial_t C_{\mathbf{k}}^{\alpha\alpha'} = \sum_{\mathbf{k}'+\mathbf{k}''=\mathbf{k}} \sum_{\beta\gamma} \Bigg[\pi |M_{\mathbf{k},\mathbf{k}',\mathbf{k}''}^{\alpha\beta\gamma}|^2 C_{\mathbf{k}'}^{\beta\beta} C_{\mathbf{k}''}^{\gamma\gamma} \delta(\omega_{\mathbf{k}}^{\alpha} - \omega_{\mathbf{k}'}^{\beta} - \omega_{\mathbf{k}''}^{\gamma}) \delta_{\alpha\alpha'}$$

$$+ M_{\mathbf{k},\mathbf{k}',\mathbf{k}''}^{\alpha\beta\gamma} M_{\mathbf{k}',\mathbf{k},-\mathbf{k}''}^{\beta\alpha\gamma} C_{\mathbf{k}}^{\alpha\alpha'} C_{\mathbf{k}''}^{\gamma\gamma} \left(\pi\delta(\omega_{\mathbf{k}}^{\alpha} - \omega_{\mathbf{k}'}^{\beta} - \omega_{\mathbf{k}''}^{\gamma}) + i\mathcal{P}\frac{1}{\omega_{\mathbf{k}}^{\alpha} - \omega_{\mathbf{k}'}^{\beta} - \omega_{\mathbf{k}''}^{\gamma}} \right)$$

$$+ M_{\mathbf{k},\mathbf{k}',\mathbf{k}''}^{\alpha'\beta\gamma *} M_{\mathbf{k}',\mathbf{k},-\mathbf{k}''}^{\beta\alpha'\gamma *} C_{\mathbf{k}}^{\alpha\alpha'} C_{\mathbf{k}''}^{\gamma\gamma} \left(\pi\delta(\omega_{\mathbf{k}}^{\alpha'} - \omega_{\mathbf{k}'}^{\beta} - \omega_{\mathbf{k}''}^{\gamma}) - i\mathcal{P}\frac{1}{\omega_{\mathbf{k}}^{\alpha'} - \omega_{\mathbf{k}'}^{\beta} - \omega_{\mathbf{k}''}^{\gamma}} \right) \Bigg].$$

where $\langle \phi_{\mathbf{k}}^{\alpha} \phi_{\mathbf{k}'}^{\alpha'} \rangle = C_{\mathbf{k}}^{\alpha\alpha'} \delta(\mathbf{k} + \mathbf{k}') e^{-i(\omega_{\mathbf{k}}^{\alpha} - \omega_{\mathbf{k}}^{\alpha'})t}$.

- ϕ^α is assumed to be an eigenmode. $M_{\mathbf{k},\mathbf{k}',\mathbf{k}''}^{\alpha\beta\gamma}$ are symmetrized coupling coefficients.
- PV integrals vanish in case of real coupling coefficients and a single field $\rightarrow$ recover classical Sagdeev-Galeev result.

Return to physical basis

- Next, specialize to β-plane MHD problem, return to familiar velocity/magnetic field basis:

$$k^2 C_{\mathbf{k}}^{\pm\pm} = \frac{1}{\Omega^2}\left(\omega_\pm^2 E_{\mathbf{k}}^K + \omega_A^2 E_{\mathbf{k}}^M - 2\omega_A\omega_\pm \operatorname{Re} H_{\mathbf{k}}\right) \tag{1}$$

$$k^2 \operatorname{Re}(C_{\mathbf{k}}^{+-} e^{-i\Omega t}) = -\frac{1}{\Omega^2}\left(\omega_A^2(E_{\mathbf{k}}^K - E_{\mathbf{k}}^M) + \omega_\beta\omega_A \operatorname{Re} H_{\mathbf{k}}\right) \tag{2}$$

$$k^2 \operatorname{Im}(C_{\mathbf{k}}^{+-} e^{-i\Omega t}) = -\frac{\omega_A}{2\Omega} \operatorname{Im} H_{\mathbf{k}}. \tag{3}$$

  where $\Omega \equiv \omega_+ - \omega_- = \sqrt{4\omega_A^2 + \omega_\beta^2}$, $E_{\mathbf{k}}^K = \langle|\tilde{v}_{\mathbf{k}}|^2\rangle$, $E_{\mathbf{k}}^M = \langle|\tilde{b}_{\mathbf{k}}|^2\rangle$, $H_{\mathbf{k}} = \langle\tilde{\mathbf{v}}_{\mathbf{k}} \cdot \tilde{\mathbf{b}}_{-\mathbf{k}}\rangle$
- Note cross-helicity will contribute to energy dynamics
- Can be solved numerically in principle. But how to make analytic progress with this mess?

MHD limit and singularity

- Weak 2D MHD previously studied [Tronko et al., 2013]. Fixed point is $E_{\mathbf{k}}^{K} = E_{\mathbf{k}}^{M} = \mathrm{const.}, H_{\mathbf{k}} = 0$
- Natural idea: use small-β perturbation theory about MHD spectra
- After some work, one finds that only $O(\beta)$ effect for $|k_x| > k_{MR}$ is to mix the turbulent energies in **k**-space
- Effect on $E_{\mathbf{k}}^{K} - E_{\mathbf{k}}^{M}$ is at least $O(\beta^3)$. That calculation remains open to the brave and bored
- The most interesting effects on the spectra are happening at small k_x, but here WT is no longer self-consistent

Cross-spectral identity

- Is WT useless then? No!
- Observe that Rossby-Alfvén cross-correlator naturally oscillates at $\omega_+ - \omega_- = \sqrt{4\omega_A^2 + \omega_\beta^2}$. On timescales longer than linear, time average is zero!
- We have again

$$k^2 \,\mathrm{Re}(C_{\mathbf{k}}^{+-} e^{-i\Omega t}) = -\frac{1}{\Omega^2}\left(\omega_A^2(E_{\mathbf{k}}^K - E_{\mathbf{k}}^M) + \omega_\beta \omega_A \,\mathrm{Re}\, H_{\mathbf{k}}\right)$$

$$\Longrightarrow \boxed{\langle E_{\mathbf{k}}^K - E_{\mathbf{k}}^M \rangle_t = \frac{\beta}{b_0 k^2} \langle \mathrm{Re}\, H_{\mathbf{k}} \rangle_t}$$

Time-averaged, stationary cross-helicity spectrum entirely determines momentum transport!

Cross-spectral identity II

- Buildup of cross-helicity during transition thus linked to breakdown of Alfvénization condition $|\tilde{v}_{\mathbf{k}}|^2 = |\tilde{b}_{\mathbf{k}}|^2$
- Can rearrange to find:

$$\frac{\langle \partial_t \tilde{v} \rangle_{\mathbf{k}}}{\langle \partial_t \tilde{b} \rangle_{\mathbf{k}}} = \frac{k_{\mathrm{MR}}^2}{k^2}.$$

$\implies$ Fluctuations kinetic for $\ell > \ell_{MR}$, magnetic for $\ell < \ell_{MR}$ [Diamond et al., 2007]

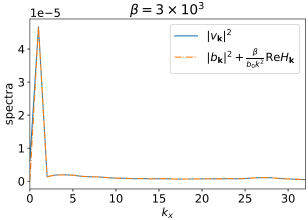


Figure Time-averaged, k_y-averaged spectra from simulation, confirming calculation. Note that spectra don't agree at $k_x = 0$ because $\Omega \to 0$

Combining with Zel'dovich

- Can integrate cross-spectral identity over $\mathbf{k}$ and combine with the CH estimate

$$H \simeq \frac{\beta}{b_0 k_0^2} \langle \tilde{b}^2 \rangle$$

to find

$$\frac{\langle \tilde{v}^2 \rangle_{\mathrm{NZ}}}{\langle \tilde{b}^2 \rangle} - 1 \sim \frac{k_{\mathrm{MR}}^4}{k_0^4}$$

for some characteristic scale k_0 (expect $\sim k_f$)

- Quantifies the degree of de-Alfvénization for $\beta / b_0 k_f^2 \lesssim 1$

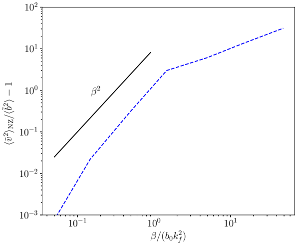


Figure Plot of $\frac{\langle \tilde{v}^2 \rangle_{\mathrm{NZ}}}{\langle \tilde{b}^2 \rangle} - 1$, compared to expected β^2 scaling

Flux of magnetic potential

- $\mathrm{Im}(C_{\mathbf{k}}^{+-} e^{-i\Omega t})$ must similarly vanish after time-averaging
- Thus $\mathrm{Im}\, H_{\mathbf{k}} \to 0 \implies \langle \tilde{v}_y \tilde{A} \rangle \to 0$.
- In other words, turbulent resistivity is zero in weak turbulence. Sufficiently strong mean field will be very long-lived
- Agrees with intuition from (e.g.) [Cattaneo and Vainshtein, 1991] – even a weak field quenches flux of A in 2D MHD. Also Zel'dovich: $\eta_T = \eta \langle \tilde{b}^2 \rangle / b_0^2$

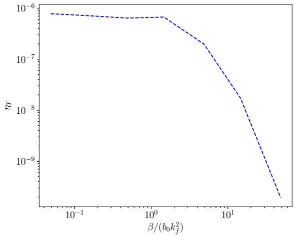


Figure Turbulent resistivity from simulation. $\eta_T \ll \eta = 10^{-4}$

Near-zonal flows

- Finally, we make the interesting observation that in the transitional regime, spectra are sharply peaked at smallest available $k_x > 0$
- Dynamics thus dominated by "near-zonal" flows, with characteristic wavelength equal to the box size. Why?
- Should study this phenomenon more carefully. Might expect something similar in drift-wave system near $\alpha = 1$. Partial suppression of transport?

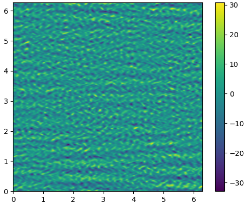


Figure Snapshot of vorticity $\nabla^2\psi$ for $\beta = 3 \times 10^3$ at $t = 400$

Spectra for $\beta = 3 \times 10^3$

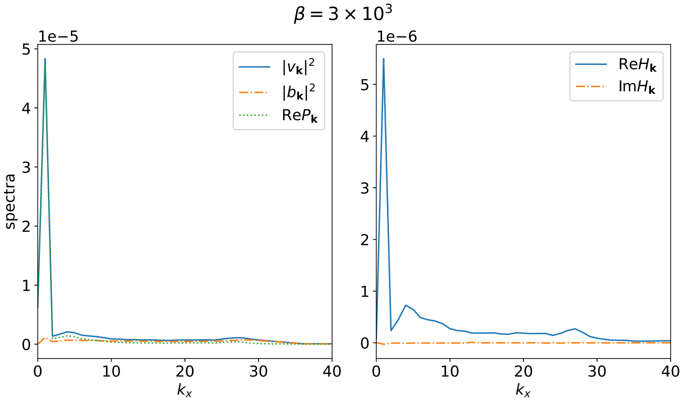


Figure Stationary spectra, averaged over k_y, for $\beta = 3 \times 10^3$.

Conclusion

- Cross helicity is non-conserved in β-plane MHD. In presence of mean magnetic field, attains a finite stationary value
- In weak turbulence theory, stationary cross-helicity spectrum equivalent to Maxwell-Reynolds stress $\rightarrow$ determines momentum transport
- Have confirmed both of these calculations in simulation
- Need strong turbulence to understand zonal flows
- $H = \frac{\beta \langle \tilde{b}^2 \rangle \ell_b \ell_v}{b_0 (1+\mathrm{Pm})}$ could be very large for weak b_0, large Rm. Should study this case numerically! Flux of magnetic potential?
- CH spectrum related to turbulent emf, but need 3D to study dynamo

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WT spectral equations sketch

- Sketch: assume ϕ^α in eigenbasis:

$$\partial_t \phi^\alpha_{\mathbf{k}} + i\omega^\alpha_{\mathbf{k}} \phi^\alpha_{\mathbf{k}} = \sum_{\beta\gamma} \frac{1}{2} \int d^2\mathbf{k}' d^2\mathbf{k}''\, \delta(\mathbf{k}-\mathbf{k}'-\mathbf{k}'') M^{\alpha\beta\gamma}_{\mathbf{k},\mathbf{k}',\mathbf{k}''} \phi^\beta_{\mathbf{k}'} \phi^\gamma_{\mathbf{k}''}, \tag{4}$$

  assume WLOG $M^{\alpha\beta\gamma}_{\mathbf{k},\mathbf{k}',\mathbf{k}''} = M^{\alpha\gamma\beta}_{\mathbf{k},\mathbf{k}'',\mathbf{k}'}$

- Use second-order time-dependent perturbation theory

$$\hat{\phi}^\alpha_{\mathbf{k}}(t) = \hat{\phi}^\alpha_{\mathbf{k}}(0) + \delta\hat{\phi}^{\alpha,(1)}_{\mathbf{k}}(t) + \delta\hat{\phi}^{\alpha,(2)}_{\mathbf{k}}(t) + \dots, \tag{5}$$

  where $\hat{\phi}^\alpha_{\mathbf{k}} = e^{i\omega^\alpha_{\mathbf{k}} t} \phi^\alpha_{\mathbf{k}}$.

- Apply random phase approx., assume spatial homeogeneity, and evaluate time integrals in limit $\omega^{-1} < t < \tau_{NL}$