Physics of Turbulence Spreading and Explicit Nonlocality

2021 AAPPS-DPP

∗This research was supported by the U.S. Department of Energy, Office of Science, Office of Fusion Energy Sciences, under Award Number DE-FG02-04ER54738.

Qinghao Yan[1,2], Patrick H. Diamond[3,2]

1. Department of Engineering Physics, Tsinghua University, Beijing 100084, PRC
2. Center for Fusion Sciences, Southwestern Institute of Physics, Chengdu, Sichuan 610041, PRC
3. Center for Astrophysics and Space Sciences (CASS) and Department of Physics, University of California San Diego, La Jolla, California 92093, USA

"Standard Model" of DW - ZF turbulence:

Disparate profile scale L_T, L_n, L_P and correlation scale Δr_c
$\Rightarrow$ local mixing, *local gradient:* $Q = -\nabla T$
$\Rightarrow D = \rho_* D_B$. $D_B = C_s \rho_*, \rho_* = \rho_i / a$.

- Breaking of gyro-Bohm $D \sim \rho_*^\sigma D_B, \sigma < 1$
- "Nonlocal phenomena"

How do turbulence and transport front propagate? Local but fast propagate? (Explicitly) non-local?

Theory Extension

- Turbulence Spreading
- Avalanching

Core idea is replacing the local Fick's law $Q = -\nabla T$ with a delocalize flux-gradient relation [1, 2, 3]

$$Q = -\int dr' K(r - r') \nabla T(r') \quad (1)$$

where $K(r - r')$ is the nonlocal kernel.

We show that, $\langle \tilde{\phi}^2 \rangle$ evolution is *explicitly* non-local. And such non-locality can affect turbulence spreading.

Explicitly Nonlocal	vs.	Heuristic Model
$\partial_t \langle \tilde{\phi}^2 \rangle = \int \gamma(r - r') \langle \tilde{\phi}^2 \rangle (r') dr' + \cdots$	vs.	$\partial_t \mathcal{E} = \partial_x [(D_0 \mathcal{E}) \partial_x \mathcal{E}] + \gamma(x) \mathcal{E} - \sigma \mathcal{E}^2$

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Roadmap

KE: $\partial_t \bar{f} + \Omega_D E \partial_\alpha \bar{f} - [J_0\phi, \bar{f}] = 0$, QuasiNeutral: $n_i = n_e$ ⟶ Darmet Model: h_i

⇣

Goal: Evolution of $\langle \tilde{\phi}^2 \rangle$

For low frequency turbulence in Tokamak ($\omega < \omega_b$, bounce frequency):

$f(\vec{r}, \vec{p}, t) \xrightarrow[Bounce-average]{Gyro-average} \bar{f}(\psi, \alpha, E, t)$. ψ radial, α angle, and E is the energy[4].

$$\begin{cases} \partial_t \bar{f} + \Omega_D E \partial_\alpha \bar{f} - [J_0 \phi, \bar{f}] = 0 \\ n_i = n_e \end{cases} \quad (2)$$

where $[F, G] = \partial_\alpha F \partial_\psi G - \partial_\psi F \partial_\alpha G$.

+

- Mean, adiabatic and non-adiabatic: $\bar{f} = \langle f \rangle - \frac{q_{i,e}\phi}{T_{i,e}} \langle f \rangle + h_{i,e}$.
- Fluctuation not response to zonal potential: $\tilde{n}_{i,e}/n_0 = -q_{i,e}(\phi - \langle \phi \rangle_\alpha)/T_{i,e}$

The non-adiabatic distribution function h_i and quasi-neutrality equation (Darmet Model [4, 5, 6]):

$$\partial_t h_i + \Omega_D E \partial_\alpha h_i - \left[\bar{\phi}, -\frac{q}{T_i}(\phi - \langle \phi \rangle_\alpha) \langle f_i \rangle + h_i \right] = \partial_t \left(\frac{q}{T_i} (\phi - \langle \phi \rangle_\alpha) \langle f_i \rangle \right) + \partial_\alpha (\overline{\phi - \langle \phi \rangle_\alpha}) \partial_\psi \langle f_i \rangle \quad (3)$$

$$C_{ad} (\phi - \langle \phi \rangle_\alpha) - C_i \overline{\Delta}_{i+e} \phi = \frac{2}{n_0 \sqrt{\pi}} \int_0^\infty J h_i \sqrt{E} dE - \frac{2}{n_0 \sqrt{\pi}} \int_0^\infty J h_e \sqrt{E} dE \quad (4)$$

where $C_i = q/T_i$, $C_{ad} = C_i(1+\tau)/\sqrt{2\varepsilon_0}$, $\tau = T_i/T_e$. $\overline{\Delta}_s = \rho_{0s}^2 \partial_\alpha^2 + \delta_{bs}^2 \partial_\psi^2$. A minimal K.S. for DW turbulence.

Roadmap

KE: $\partial_t \bar{f} + \Omega_D E \partial_\alpha \bar{f} - [J_0\phi, \bar{f}] = 0$, QuasiNeutral: $n_i = n_e$ ⟶ Darmet Model: h_i

Moment / Separate by symmetry ⟶ Evolution of Fluctuation $\tilde{T}$、$\tilde{U} \equiv \tilde{\phi} - \overline{\Delta}\tilde{\phi}$

⇢ Goal: Evolution of $\langle\tilde{\phi}^2\rangle$

$h_e = 0$ and neglect $\overline{\Delta}_e$. Taking the derivative of equation (4) w.r.t. time. Separate the results according to symmetry in angle direction. $\phi = \tilde{\phi} + \phi_Z$[7, 8].

$$\left(\frac{\partial}{\partial t} + \widetilde{\mathbf{V}}\cdot\nabla + \mathbf{V}_Z\cdot\nabla\right)(C_i\overline{\Delta}\widetilde{\phi}) = \frac{3}{2}\Omega_D\partial_\alpha\widetilde{T}_i - iC_e(\omega - \omega_E + \frac{\omega^i_{*n}}{\tau})\widetilde{\phi} - C_i\widetilde{V}(\psi)\partial_\psi(\overline{\Delta}\phi_Z) \tag{5}$$

$$\frac{\partial}{\partial t}\left[C_i\overline{\Delta}\phi_Z\right] = C_i\langle\nabla\widetilde{\phi}\times\hat{z}\cdot(\nabla\overline{\Delta}\widetilde{\phi})\rangle_\alpha \equiv -C_i\delta^2_{b0}\partial^2_\psi\left\langle\tilde{v}_\psi\tilde{v}_\alpha\right\rangle_\alpha \tag{6}$$

Defined *potential-vorticity* quantity: $\tilde{U} \equiv C_e\tilde{\phi} - C_i\overline{\Delta}\tilde{\phi}$. Then:

$$\text{Eq.}(5) \Longrightarrow \left(\frac{\partial}{\partial t} + \widetilde{\mathbf{V}}\cdot\nabla + \mathbf{V}_Z\cdot\nabla\right)\tilde{U} = -\frac{3}{2}\Omega_D\partial_y\widetilde{T}_i + C_i\widetilde{V}(r)\partial_r(\overline{\Delta}\phi_Z) \tag{7}$$

where $(\psi, \alpha) \rightarrow \vec{x} \equiv (r, y)$, Ω_D is a typical (constant) ion precession velocity. Equation above is similar to the H-M eq. Potential vorticity $\tilde{U}$ is a *conserved macro-quantity*, here broken by the linear terms.

Potential vorticity conservation equation:

$$\left(\frac{\partial}{\partial t}+\tilde{\mathbf{V}}\cdot\nabla+\mathbf{V}_Z\cdot\nabla\right)\tilde{U}=-\frac{3}{2}\Omega_D\partial_y\tilde{T}_i+C_i\tilde{V}(r)\partial_r(\overline{\Delta}\phi_Z) \quad (7)$$

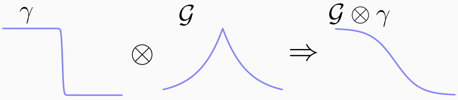


Fig 1: Effect of convolution with $\mathcal{G}$

$\tilde{U}\Rightarrow\tilde{\phi}$?

According to the definition, there is

$$\tilde{U}_{\bar{k}}=(C_e+C_i\bar{k}^2)\tilde{\phi}_{\bar{k}}\longrightarrow\tilde{\phi}_{\bar{k}}=\frac{\tilde{U}_{\bar{k}}}{C_e+C_i\bar{k}^2}$$

$$\xrightarrow{\text{yields}}\tilde{\phi}=\int\mathcal{G}(x,x')\tilde{U}(x')dx'\equiv\mathcal{G}\otimes\tilde{U} \quad (8)$$

where Green's function:

$$\mathcal{G}(x,x')=\frac{\sqrt{A}}{2}e^{-\sqrt{A}|x-x'|},\quad A\sim\delta_b^{-2} \quad (9)$$

Naturally, the intensity of $\langle\tilde{\phi}^2\rangle$ is:

$$\left\langle\tilde{\phi}^2\right\rangle=\lim_{1\to2}\iint G(x_1,x_1')G(x_2,x_2')\left\langle\tilde{U}(x_1')\tilde{U}(x_2')\right\rangle dx_1'dx_2'$$

Terms like $\left\langle\tilde{v}_{r_1}\widetilde{U_1U_2}\right\rangle$ can be closed by *two-point quasilinear approximation*,

$$\left(\widetilde{U_1U_2}\right)_{\substack{k_y\\ \omega}}=-\left[R^{(1)}_{\substack{k_y\\ \omega}}\tilde{v}_{\substack{k_y\\ r}}(x_1)e^{ik_yy_1}\partial_{r_1}+R^{(1)}_{\substack{k_y\\ \omega}}\tilde{v}_{\substack{k_y\\ y}}(x_1)e^{ik_yy_1}\partial_{y_1}\right.$$
$$\left.+R^{(2)}_{\substack{k_y\\ \omega}}\tilde{v}_{\substack{k_y\\ r}}(x_2)e^{ik_yy_2}\partial_{r_2}+R^{(2)}_{\substack{k_y\\ \omega}}\tilde{v}_{\substack{k_y\\ y}}(x_2)e^{ik_yy_2}\partial_{y_2}\right]\left\langle\tilde{U}_1\tilde{U}_2\right\rangle$$

Roadmap

KE: $\partial_t \bar{f} + \Omega_D E \partial_\alpha \bar{f} - [J_0 \phi, \bar{f}] = 0$, QuasiNeutral: $n_i = n_e$ → Darmet Model

Moment

Separate by symmetry

Evolution of Fluctuation $\tilde{T}$、$\tilde{U} \equiv \tilde{\phi} - \overline{\Delta \tilde{\phi}}$

2-p correlation [9]

$\langle \tilde{U}(1) \partial_t \tilde{U}(2) + \tilde{U}(2) \partial_t \tilde{U}(1) \rangle$

Evolution of PE: $\langle \tilde{U}(1) \tilde{U}(2) \rangle$

Goal: Evolution of $\langle \tilde{\phi}^2 \rangle$

Roadmap

KE: $\partial_t\bar{f} + \Omega_D E\partial_\alpha\bar{f} - [J_0\phi, \bar{f}] = 0$, QuasiNeutral: $n_i = n_e$ → Darmet Model

Moment / Separate by symmetry →

Evolution of Fluctuation $\tilde{T}$、$\tilde{U} \equiv \tilde{\phi} - \overline{\Delta}\tilde{\phi}$

2-p correlation: $\langle\tilde{U}(1)\partial_t\tilde{U}(2) + \tilde{U}(2)\partial_t\tilde{U}(1)\rangle$ →

Evolution of PE: $\langle\tilde{U}(1)\tilde{U}(2)\rangle$

2-p quasilinear method / Green's function →

Goal: Evolution of $\langle\tilde{\phi}^2\rangle$

The evolution equation of potential intensity [10]:

$$\frac{\partial}{\partial t}\left\langle\tilde{\phi}^2\right\rangle = \mathcal{G}\otimes\frac{\partial}{\partial r}\left[2D_0\left\langle\tilde{\phi}^2\right\rangle\frac{\partial}{\partial r}\left(\left\langle\tilde{\phi}^2\right\rangle - \frac{\delta_b^2}{2}\frac{\partial^2}{\partial r^2}\left\langle\tilde{\phi}^2\right\rangle\right)\right] + \mathcal{G}\otimes\left(\gamma_L(r)\left\langle\tilde{\phi}^2\right\rangle\right) - \frac{D_0}{l_r^2}\left\langle\tilde{\phi}^2\right\rangle^2 \quad (10)$$

Heat flux drive approximated: $\langle\tilde{v}_r\tilde{T}\rangle \sim -\langle\tilde{\phi}^2\rangle\partial_r\langle T\rangle \sim -\gamma_L\langle\tilde{\phi}^2\rangle$ (assumed $\partial_r\langle T\rangle \sim \langle T\rangle/L_T > 0$).
Neglected the ϕ_Z for simplicity.

- Nonlocal nonlinear diffusion: Nonlocality is weak as shown latter, simplified as $\partial_r(2D_0\langle\tilde{\phi}^2\rangle\partial_r\langle\tilde{\phi}^2\rangle)$
- Nonlocal growth: *Distributed pumping of $\langle\tilde{\phi}^2\rangle$ from the heat flux $\langle\tilde{v}_r\tilde{T}\rangle$.*

  Kernel width of $\mathcal{G}(x,x') \propto \exp(-|x-x'|/\delta_b)$ is several δ_b, thus the growth of $\langle\tilde{\phi}^2\rangle$ at r is affected by a region of several δ_b in width. Preconditions:

  1. The curvature of the field $\Rightarrow$ trapped ion orbit and ion-precessional motion.
  2. The polarization charge due to trapped ions $\Rightarrow$ redistribution of fluctuating temperature.

- Nonlinear local damping: $D_{y,y} \approx 2\sum_{k_y,\omega} R_{k_y}\left|\tilde{\phi}_k\right|^2\frac{k_y^2}{k_y^2 l_r^2}(1-\cos(k_y y_-)) \xrightarrow{\langle y_-^2\rangle > 1} \approx 2D_0\left\langle\tilde{\phi}^2\right\rangle\frac{1}{k_y^2 l_r^2}$

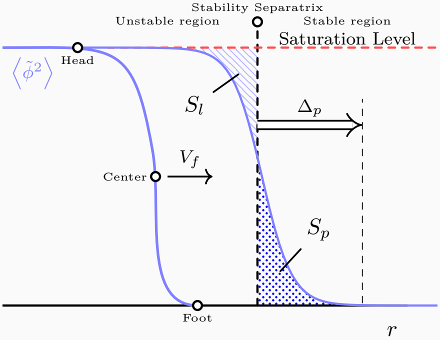
Spreading Model

Heuristic Model[11, 12] vs. Explicitly Nonlocal Model

$$\partial_t \mathcal{E} = \partial_x[(D_0\mathcal{E})\partial_x\mathcal{E}] + \gamma(x)\mathcal{E} - \sigma\mathcal{E}^2 \quad \text{vs.} \quad \partial_t \left\langle \tilde{\phi}^2 \right\rangle = \mathcal{G} \otimes \text{N-lin. Diff.} + \mathcal{G} \otimes \left(\gamma_L(r)\left\langle \tilde{\phi}^2 \right\rangle\right) - \frac{D_0}{I_r^2}\left\langle \tilde{\phi}^2 \right\rangle^2$$

Illustration of quantities:

- V_f, the leading edge propagating speed
- Shape of front characterized with distance between "Foot", "Center" and "Head"
- Penetration of leading edge into the stable region:
  - Depth, Δ_p
  - Area, S_p



How do those nonlocal terms affect spreading front generation and propagation?

Wider, Faster and Deeper

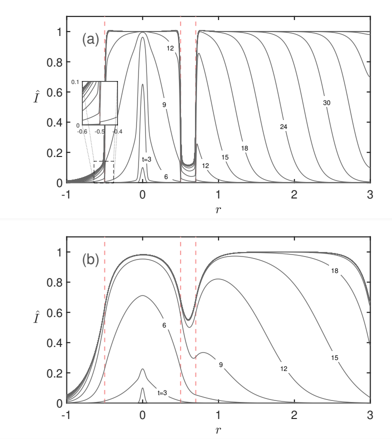


Fig 2: Evolution of (a) with nonlocal diffusion, (b) with nonlocal growth.

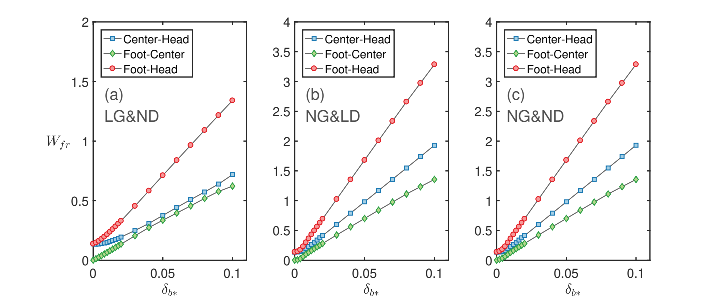


Fig 3: Width of the propagating front in different equations with a fixed ρ_i when varying δ_b.

- $W_f \propto \delta_b$
- $\mathcal{G} \otimes (\gamma_L(r)\langle\tilde{\phi}^2\rangle)$ is much more effective.

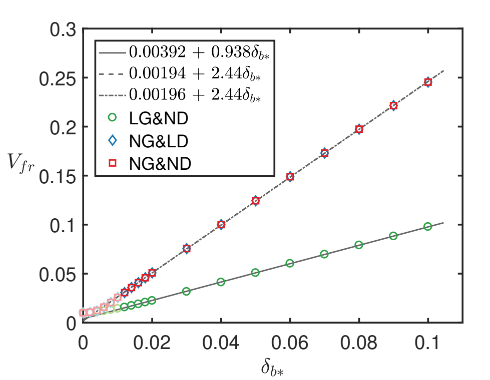


Fig 4: Leading edge propagation speed for different models when varying δ_b with $\rho_i = 0.01$. Data points with lighter colors indicate where $\delta_b < \rho_i$ and are excluded from the fit lines.

- $\delta_b \to 0$, the speed converges to classic Fisher-KPP front speed $\sqrt{2\gamma D} = 0.01$ [12].
- $\delta_b > \rho_i$, $V_f \propto \sqrt{2\gamma D}(1 + \delta_b)$
- Data form NG&ND and NG&LD overlapping indicates that the nonlocal growth effect dominates.

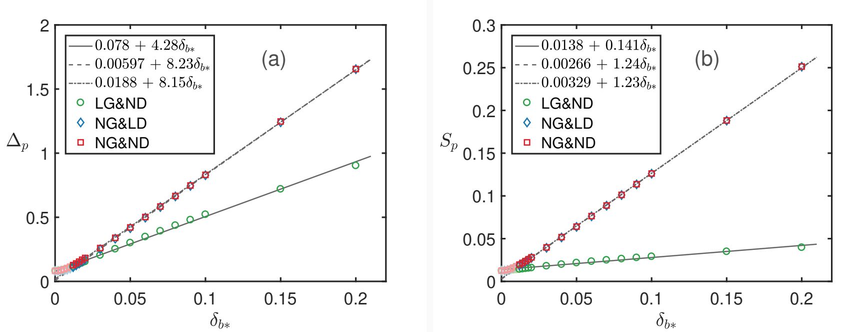


Fig 5: Front penetration Δ_p (a) and effective penetration S_p (b) against δ_{b*} for different equations. Simple linear relation can fit both Δ_p and S_p, when $\delta_{b*} > \rho_*$. Data points in lighter colors are excluded from the fits.

$$\Delta_p, S_p \propto \delta_{b*} \xrightarrow[\text{Domain}]{\text{Symmetric}} \bar{D}(\langle\tilde{\phi}^2\rangle) \propto 1 - S_l = 1 - \delta_{b*} \tag{11}$$

where $\delta_{b*} = \delta_b / L_T$.

$$\text{ZF: } \frac{\partial}{\partial t}\left[\overline{\Delta}\phi_Z\right] = -\partial_r \left\langle \tilde{v}_r \overline{\Delta}\tilde{\phi} \right\rangle_y + \nu \frac{\partial^2}{\partial r^2}\overline{\Delta}\phi_Z$$

$$\text{PE: } \frac{1}{2}\frac{\partial}{\partial t}\langle U^2\rangle = \frac{\partial}{\partial r} D^K \frac{\partial}{\partial r}\langle U^2\rangle - \bar{\Omega}_D \Re\left\{\langle \tilde{T}(A-\overline{\Delta})\tilde{v}_r\rangle\right\} - \delta_b^2 \partial_r^3 \phi_Z \left\langle \tilde{v}_r \overline{\Delta}\tilde{\phi}\right\rangle_y$$

Zonal Energy and PE are connected through vorticity flux

$$\begin{cases} \int \overline{\Delta}\phi_Z \frac{\partial}{\partial t}\left[\overline{\Delta}\phi_Z\right] \mathrm{d}r \Longrightarrow \int \delta_b^2 \partial_r^3 \phi_Z \left\langle \tilde{v}_r \overline{\Delta}\tilde{\phi}\right\rangle_y \mathrm{d}r + \cdots \\ \int \frac{1}{2}\frac{\partial}{\partial t}\langle U^2\rangle \mathrm{d}r \Longrightarrow -\int \delta_b^2 \partial_r^3 \phi_Z \left\langle \tilde{v}_r \overline{\Delta}\tilde{\phi}\right\rangle_y \mathrm{d}r + \cdots \end{cases}$$

$$\langle \tilde{v}_r \overline{\Delta}\tilde{\phi}\rangle_y \sim \chi_1^{\mathrm{non-res}} \frac{\partial_r \ln\langle T\rangle}{\sqrt{2\varepsilon_0}} - \left(\chi_2^{\mathrm{non-res}} + \chi_2^{\mathrm{res}}\right) \delta_b^2 \partial_r^3 \phi_Z$$

$$\chi_2^{\mathrm{res}} \sim \sum_k \left[\tilde{v}_r(k)\right]^2 \pi\delta(\omega - k_y V_Z - k_y b_k \bar{\Omega}_D)$$

Ideas:

- Trace of R-K type inst.
- Resonant transport is crucial.
- $\langle T\rangle$ evolution.

Conclusions and Discussions

Roadmap

KE & QuasiNeutrality $\longrightarrow$ Darmet Model $\longrightarrow$ $\tilde{T}$、$\tilde{U} \equiv \tilde{\phi} - \overline{\Delta}\tilde{\phi}$ $\longrightarrow$ $\langle \tilde{U}(1)\tilde{U}(2)\rangle$ $\longrightarrow$ Green's function $\longrightarrow$

$$\partial_t \left\langle \tilde{\phi}^2 \right\rangle = \partial_r \left[D_0 \left\langle \tilde{\phi}^2 \right\rangle \partial_r \left\langle \tilde{\phi}^2 \right\rangle \right] + \mathcal{G} \otimes \left(\gamma_L(r) \left\langle \tilde{\phi}^2 \right\rangle \right) - \frac{D_0}{l_r^2} \left\langle \tilde{\phi}^2 \right\rangle^2$$

Conclusions

1. $\partial_t \langle \tilde{\phi}^2 \rangle$ is *explicitly nonlocal*.
2. *Explicit non-local growth* is the principal new effect.
3. Potential vorticity $\tilde{U} = A\tilde{\phi} - \overline{\Delta}\tilde{\phi}$ conservation.
4. Inverting PV to $\tilde{\phi} \Rightarrow$ Green's Function:
   $\mathcal{G}(x, x') \propto \sqrt{A} e^{-\sqrt{A}|x-x'|}$
   $\Rightarrow \delta_b$ sets range of nonlocality, which is modest.
5. $V_f \simeq (\gamma D)^{1/2}(1 + \delta_b)$, $\Delta_p \propto \delta_{b*}$

Discussions and Future Plans

- The utility of PV (potential vorticity).
- Near macro-marginality $\Rightarrow$ Explicit nonlocality $\uparrow$.
- Pedestal $\Rightarrow \delta_b / L_T \uparrow$.
- Energetic particle-driven turbulence $\Rightarrow \delta_b \uparrow$.
- Including ZF

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Thanks!

$$\left(\frac{\partial}{\partial t}+\tilde{\mathbf{V}}\cdot\nabla+\mathbf{V}_Z\cdot\nabla\right)(C_i\overline{\Delta}\tilde{\phi})=\frac{3}{2}\Omega_D\partial_\alpha\tilde{T}_i-iC_e(\omega-\omega_E+\frac{\omega^i_{*n}}{\tau})\tilde{\phi}-C_i\tilde{V}(\psi)\partial_\psi(\overline{\Delta}\phi_Z) \tag{12}$$

$$\frac{\partial}{\partial t}\left[C_i\overline{\Delta}\phi_Z\right]=C_i\langle\nabla\tilde{\phi}\times\hat{z}\cdot(\nabla\overline{\Delta}\tilde{\phi})\rangle_\alpha\equiv-C_i\delta^2_{b0}\partial^2_\psi\left\langle\tilde{v}_\psi\tilde{v}_\alpha\right\rangle_\alpha \tag{13}$$

Taking moment of (3), then obtain the fluctuating temperature $\tilde{T}_i$ evolution

$$\sqrt{2\varepsilon_0}\left(\partial_t+\tilde{\mathbf{V}}\cdot\nabla+\mathbf{V}_Z\cdot\nabla\right)\tilde{T}_i=-i(\omega-\omega_E-\omega^i_{*n}-\omega^i_{*T})\langle T_i\rangle\frac{q\tilde{\phi}}{T_i} \tag{14}$$

$$\begin{aligned}\frac{\partial}{\partial t}\left(\langle T_i\rangle+\langle T_i\rangle\ln\langle n_i\rangle\right)&=\sqrt{2\varepsilon_0}\langle\nabla\tilde{\phi}\times\hat{z}\cdot\nabla\tilde{T}_i\rangle_\alpha\\&\equiv-\sqrt{2\varepsilon_0}\partial_\psi\langle\tilde{v}_\psi\tilde{T}_i\rangle_\alpha\end{aligned} \tag{15}$$

(5) to (15) are equivalent to a set of PV (Potential Vorticity) conservation equation:

$$\partial_t\langle q\rangle = \left\langle \nabla\widetilde{\phi}\times\hat{z}\cdot\nabla\delta q\right\rangle_\alpha = -\partial_\psi\left\langle\tilde{v}_\psi\delta q\right\rangle_\alpha \tag{16}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\delta q = -\frac{3}{2}\Omega_D\partial_\alpha\widetilde{T}_i - \widetilde{\mathbf{V}}\cdot\nabla\langle q\rangle \tag{17}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + \widetilde{\mathbf{V}}\cdot\nabla + \mathbf{V}_Z\cdot\nabla, \qquad \langle q\rangle = \frac{\tau}{\sqrt{2\varepsilon_0}}\ln\langle T_i\rangle - C_i\overline{\Delta}\phi_Z, \qquad \delta q = \tau\frac{\widetilde{T}_i}{\langle T_i\rangle} - C_i\overline{\Delta}\widetilde{\phi}$$

Multiplying eq (17) with δq to obtain PE (Potential Enstrophy), and averaging in α:

$$\frac{1}{2}\partial_t\langle\delta q^2\rangle + \frac{1}{2}\partial_\psi\langle\widetilde{v}_\psi\delta q^2\rangle = -\frac{3}{2}\Omega_D\langle\delta q\partial_\alpha\widetilde{T}_i\rangle - \langle\widetilde{v}_\psi\delta q\rangle\partial_\psi\langle q\rangle \quad (18)$$

The quasilinear expression of δq is as below. Assumed $\omega(k) = \omega_R(k) + i\gamma(k)$, where $\gamma(\pm k) \geq 0$:

$$\begin{aligned}\delta q_k &= \frac{3}{2}\frac{k_\alpha\Omega_D}{\omega - k_\alpha v_Z}\widetilde{T}_k - \frac{i\partial_\psi\langle q\rangle}{\omega - k_\alpha v_Z}\widetilde{v}_\psi(k) \\ &= \frac{3}{2}\frac{k_\alpha\Omega_D[\omega_R - k_\alpha v_Z - i\gamma]}{|\omega - k_\alpha v_Z|^2}\widetilde{T}_k - \frac{i\partial_\psi\langle q\rangle[\omega_R - k_\alpha v_Z - i\gamma]}{|\omega - k_\alpha v_Z|^2}\widetilde{v}_\psi(k)\end{aligned} \quad (19)$$

Define $\langle\delta q^2\rangle/2 \equiv \mathcal{E}$, then:

$$\partial_t\mathcal{E} = \partial_\psi D_\mathcal{E}\partial_\psi\mathcal{E} + \chi_1\left[\partial_\psi\langle q\rangle\right]^2 + \chi_2\left[\frac{3}{2}\Omega_D\right]^2 - 2\chi_3\left[\frac{3}{2}\Omega_D\partial_\psi\langle q\rangle\right] \quad (20)$$

$$\chi_1 = \sum_k\frac{\gamma\langle\widetilde{v}_\psi^2\rangle_k}{|\omega - k_\alpha v_Z|^2}, \qquad \chi_2 = \sum_k\frac{\gamma k_\alpha^2\langle\widetilde{T}^2\rangle_k}{|\omega - k_\alpha v_Z|^2}, \qquad \chi_3 = \sum_k\frac{\gamma k_\alpha\Im\left\{\langle\widetilde{v}_\psi\widetilde{T}\rangle_k\right\}}{|\omega - k_\alpha v_Z|^2}$$

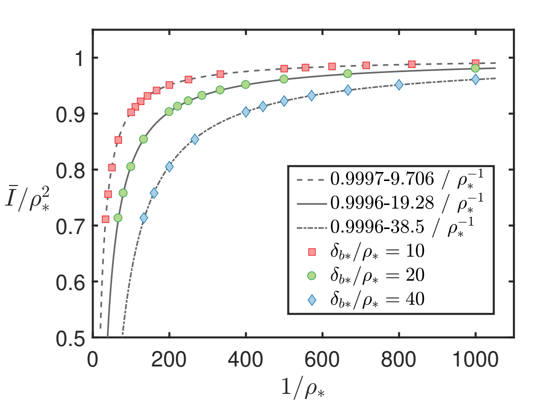

Fig 6: Transport coefficient $D = D_0 I$ transit from "D_B" to D_{GB}. A simple linear relation is obtained: $\bar{\hat{I}} = \bar{I}/\rho_*^2 = 1 - \delta_{b*}$.

Usual spreading:

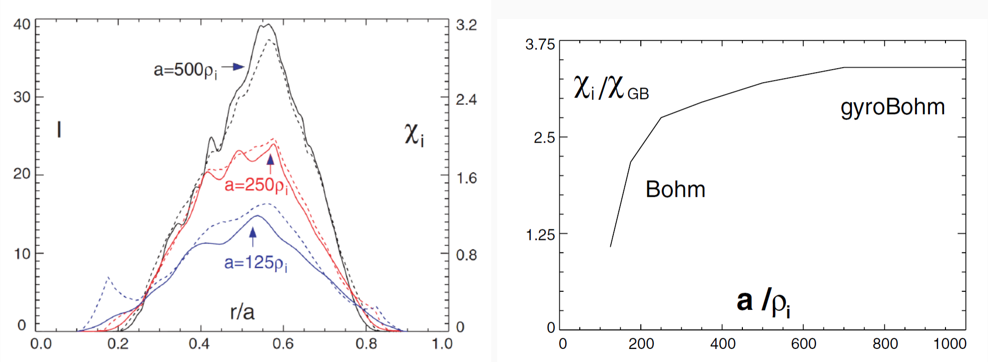

Fig 7: In GTC simulations, χ_i and turbulence intensity varies with device size[?].

$$\chi_i \simeq \frac{\chi_0}{(1+50\rho_*)^2} \tag{21}$$

"Hacking" of Mathematica©

$$\text{NDSolve} \Rightarrow \begin{cases} \text{state = NDSolve`ProcessEquations} \Leftarrow \textit{Parallelizing} \\ \text{NDSolve`Iterate[state, 0, tmax]} \\ \text{sol = NDSolve`ProcessSolutions[state]} \end{cases}$$