On How Geometry Constrains Plasma Dynamics

Mingyun Cao

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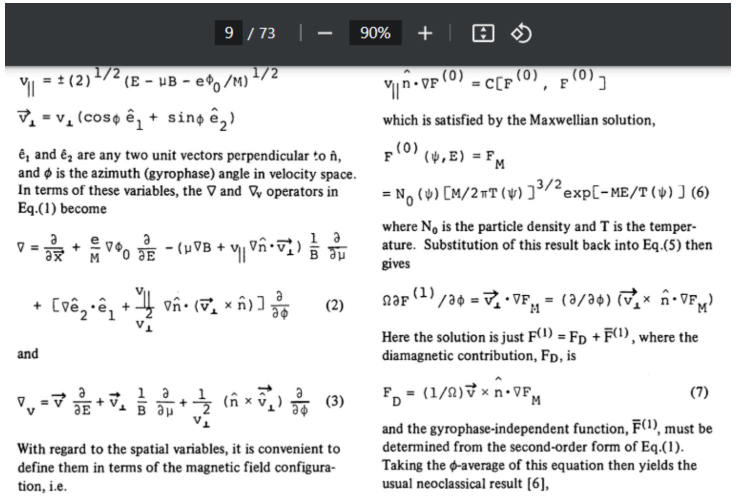
Anti-diffusive dynamics: from sub-cellular to astrophysical scales

Isaac Newton Institute for Mathematical Sciences

Before I start…

- My own work: plasma dynamics in a stochastic magnetic field
  - Motivated by the use of resonant magnetic perturbation in tokamaks
- A bit specialized… ⇒ would like to show something more accessible
- Multiple sources: Prof. Diamond's notes and other "ancient" papers.
- Unearth the physical insights buried in algebra.



$$v_{||} = \pm(2)^{1/2}(E - \mu B - e\Phi_0/M)^{1/2}$$

$$\vec{v}_\perp = v_\perp(\cos\phi\,\hat{e}_1 + \sin\phi\,\hat{e}_2)$$

$\hat{e}_1$ and $\hat{e}_2$ are any two unit vectors perpendicular to $\hat{n}$, and ϕ is the azimuth (gyrophase) angle in velocity space. In terms of these variables, the ∇ and ∇_v operators in Eq.(1) become

$$\nabla = \frac{\partial}{\partial \vec{x}} + \frac{e}{M}\nabla\Phi_0\frac{\partial}{\partial E} - (\mu\nabla B + v_{||}\nabla\hat{n}\cdot\vec{v}_\perp)\frac{1}{B}\frac{\partial}{\partial\mu}$$

$$+ [\nabla\hat{e}_2\cdot\hat{e}_1 + \frac{v_{||}}{v_\perp^2}\nabla\hat{n}\cdot(\vec{v}_\perp\times\hat{n})]\frac{\partial}{\partial\phi} \qquad (2)$$

and

$$\nabla_v = \vec{v}\frac{\partial}{\partial E} + \vec{v}_\perp\frac{1}{B}\frac{\partial}{\partial\mu} + \frac{1}{v_\perp^2}(\hat{n}\times\vec{\hat{v}}_\perp)\frac{\partial}{\partial\phi} \qquad (3)$$

With regard to the spatial variables, it is convenient to define them in terms of the magnetic field configuration, i.e.

$$v_{||}\hat{n}\cdot\nabla F^{(0)} = C[F^{(0)}, F^{(0)}]$$

which is satisfied by the Maxwellian solution,

$$F^{(0)}(\psi, E) = F_M$$

$$= N_0(\psi)[M/2\pi T(\psi)]^{3/2}\exp[-ME/T(\psi)] \quad (6)$$

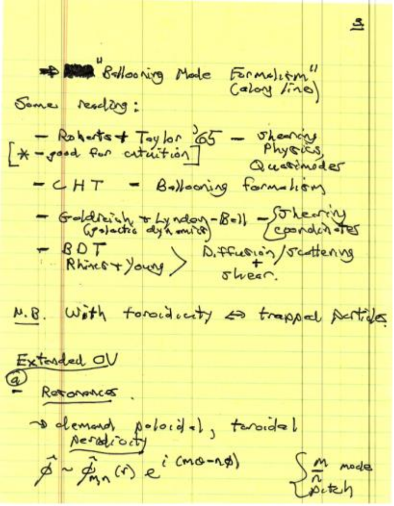
where N_0 is the particle density and T is the temperature. Substitution of this result back into Eq.(5) then gives

$$\Omega\partial F^{(1)}/\partial\phi = \vec{v}_\perp\cdot\nabla F_M = (\partial/\partial\phi)(\vec{v}_\perp\times\hat{n}\cdot\nabla F_M)$$

Here the solution is just $F^{(1)} = F_D + \bar{F}^{(1)}$, where the diamagnetic contribution, F_D, is

$$F_D = (1/\Omega)\vec{v}\times\hat{n}\cdot\nabla F_M \qquad (7)$$

and the gyrophase-independent function, $\bar{F}^{(1)}$, must be determined from the second-order form of Eq.(1). Taking the ϕ-average of this equation then yields the usual neoclassical result [6],



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⇒ [redacted] "Ballooning Mode Formalism"
(along line)

Some reading:

- Roberts + Taylor '65 – shearing physics, Quasimodes
[* – good for intuition]
- CHT – Ballooning formalism
- Goldreich + Lynden-Bell (galactic dynamics) – {shearing coordinates
- BDT
Rhines + Young 〉 Diffusion/scattering + shear.

N.B. With toroidicity ⇔ trapped particles

Extended OV

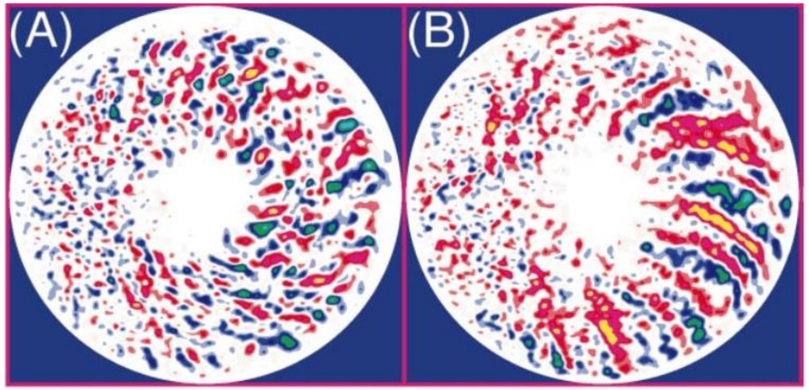
ⓐ

- Resonances.

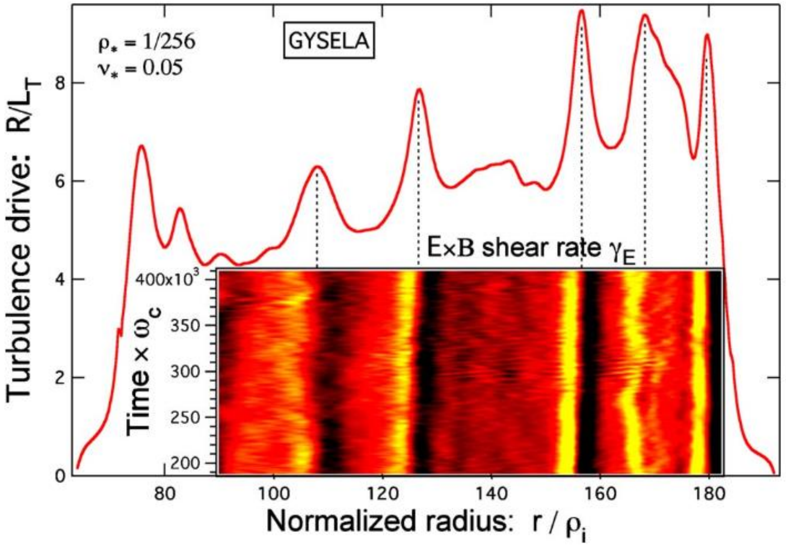
→ demand poloidal, toroidal periodicity

$\tilde{\phi} \sim \tilde{\phi}_{mn}(r)\, e^{i(m\theta - n\phi)}$ $\Big\{\frac{m}{n}$ mode pitch

Background



Lin 1998



Dif-Pradalier
2010

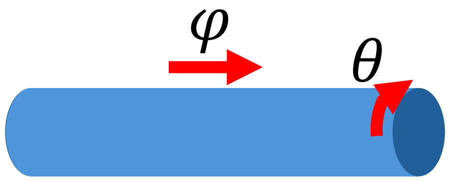
- We have seen colorful visualizations of plasma turbulence, zonal flow and staircases.
- We are also familiar with simple models, e.g.,

$$\partial_t(\nabla^2\varphi - \varphi) - [(\nabla\varphi \times \hat{\boldsymbol{z}}) \cdot \nabla][\nabla^2\varphi - \ln n_0] = 0$$

- There is a **gap** between **complex reality** and well-known **simple models**.
- To see how geometry constrains plasma dynamics → theories involving geometric effects.
- Geometry: magnetic configuration (shear & toroidicity)

Outline

Periodic cylinder



Resonant surfaces
- Definition
- Resonance 'pins' turbulence

Magnetic shear
- Line bending
- Stabilization & localization

Resistivity
- Breaking frozen-in law
- Resistive interchange

Wave-packet
- Twisted slicing coordinates
- Bridge to toroidal world

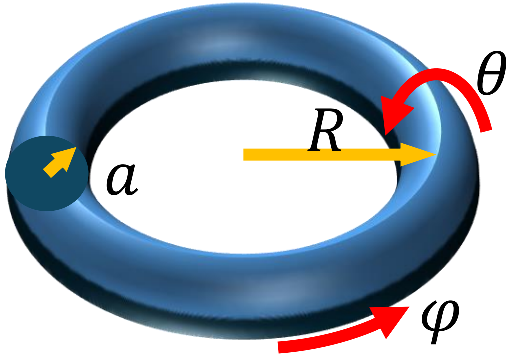
Torus

Ballooning mode
- Bloch eigenmode equation
- Ballooning mode representation

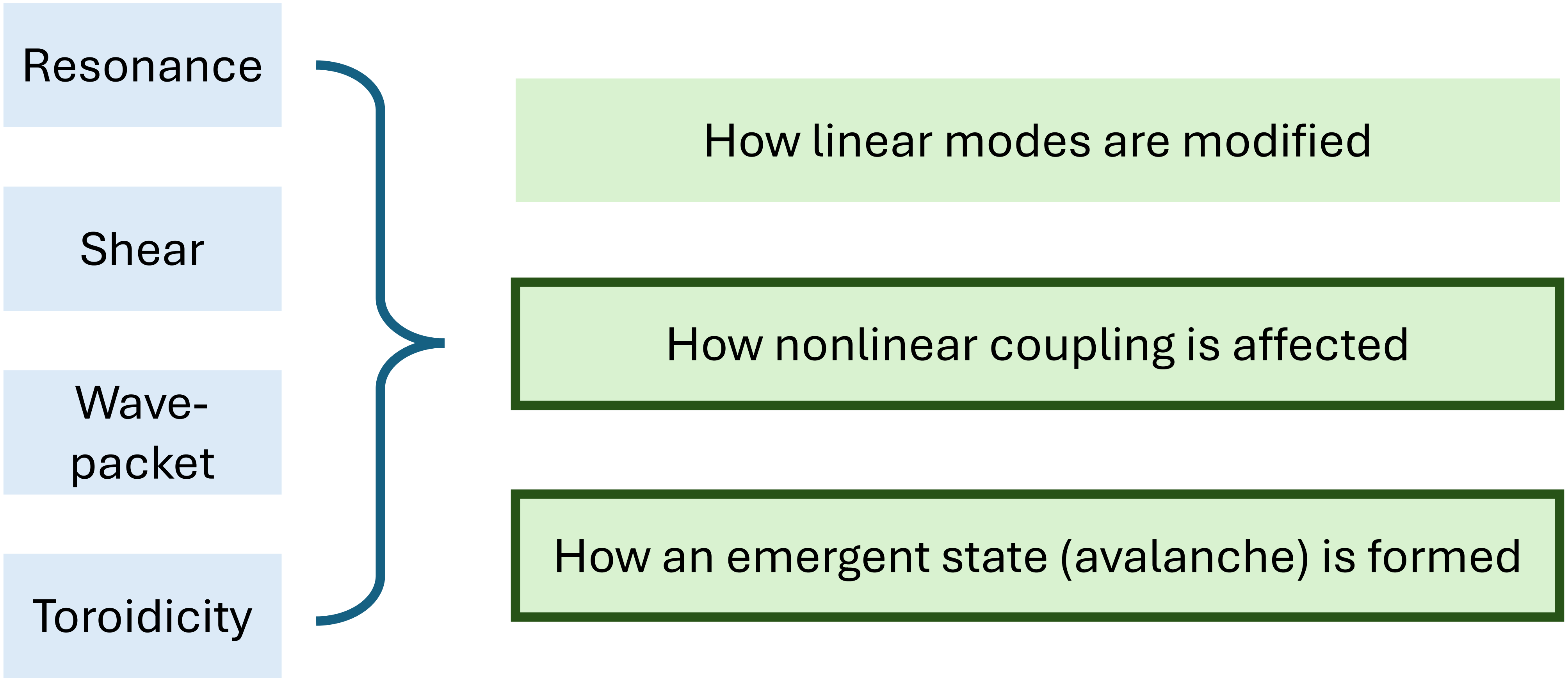
Beyond linear theory
- Shear enhanced decorrelation
- Nonlinear coupling
- Indication for avalanche

Focus on MHD

but generic



Outline: the message you can get



Resonance

Shear

Wave-packet

Toroidicity

How linear modes are modified

How nonlinear coupling is affected

How an emergent state (avalanche) is formed

Side story: velocity shear

Advertisements: my own work

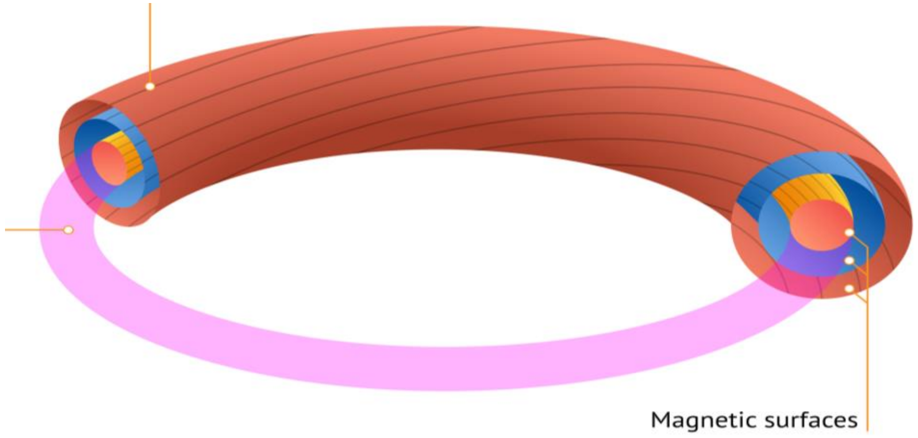
Resonances

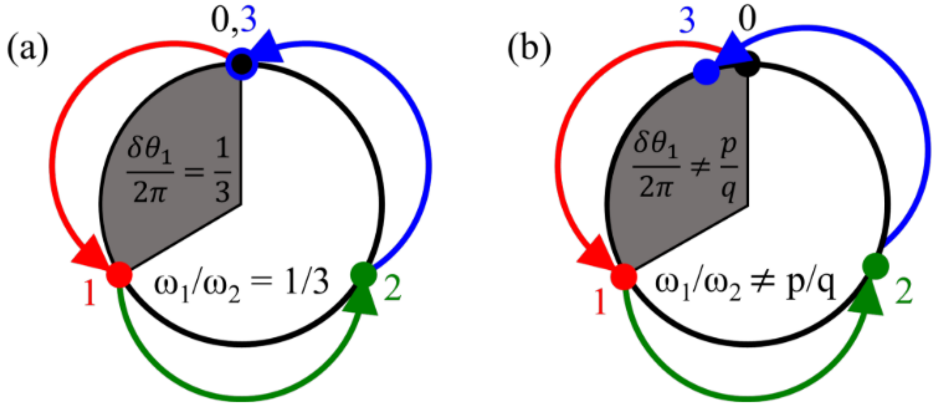
- Poloidal and toroidal periodicity ⇒ $\varphi \sim \hat{\varphi}_{m,n} \exp i(m\theta - n\phi)$ $\frac{d\phi}{d\theta} = \frac{m}{n}$ = mode pitch
- Rotational transform in tokamak:

magnetic shear

$$r d\theta / B_\theta = R d\phi / B_\phi \Rightarrow \boxed{q(r)} = d\phi/d\theta = rB_\phi / RB_\theta(r) = \text{ field line pitch}$$

- When field line pitch = mode pitch, i.e., $q(r_{m,n}) = m/n$ ⇒ resonant surfaces
- At resonant surfaces: $k_\parallel = \boldsymbol{k} \cdot \widehat{\boldsymbol{b}} = \left(m/q(r_{m,n}) - n\right)/R = 0$





resonant surface   non-resonant surface

Resonances

- Properties of $k_{\parallel} = 0$:
  - minimize Landau damping as $k_{\parallel} \to 0, v_p = \omega/k_{\parallel} \to \infty \neq v_{\parallel}$
  - minimize line bending (energetically unfavored)

$$\delta W = \frac{1}{2}\int d^3x \left\{\frac{\widetilde{\boldsymbol{B}}^2}{4\pi} + \boldsymbol{J_0}\cdot\left(\boldsymbol{\xi}\times\widetilde{\boldsymbol{B}}\right) + \gamma p_0(\nabla\cdot\boldsymbol{\xi})^2 + (\boldsymbol{\xi}\cdot\nabla p_0)(\nabla\cdot\boldsymbol{\xi}) - (\boldsymbol{\xi}\cdot\nabla\phi)\nabla\cdot(\rho_0\boldsymbol{\xi})\right\}$$

$$\widetilde{\boldsymbol{B}} = \nabla\times(\boldsymbol{\xi}\times\boldsymbol{B_0}) \cong \boldsymbol{B_0}\cdot\nabla\boldsymbol{\xi} \to \widetilde{\boldsymbol{B}}^2/4\pi \propto B_0^2 k_{\parallel}^2 > 0$$

- Resonant surfaces are **ideal "habitats"** for modes, islands, etc.
- Resonance '**pins**' turbulence. (modes live on resonant surfaces)

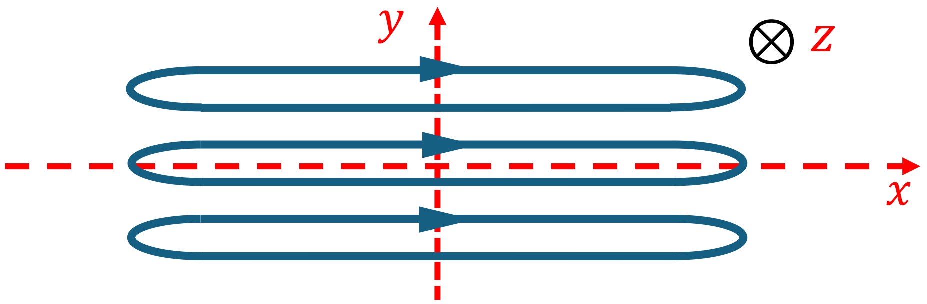
Ideal interchange mode

- Magnetic shear is present in tokamaks. **What are effects of magnetic shear on modes?**
- Start with the simplest mode: interchange (heavy pressing light)
- In the case of no shear, no resistivity, perfect alignment ⇒ $k_\parallel = 0$ (flute instability)

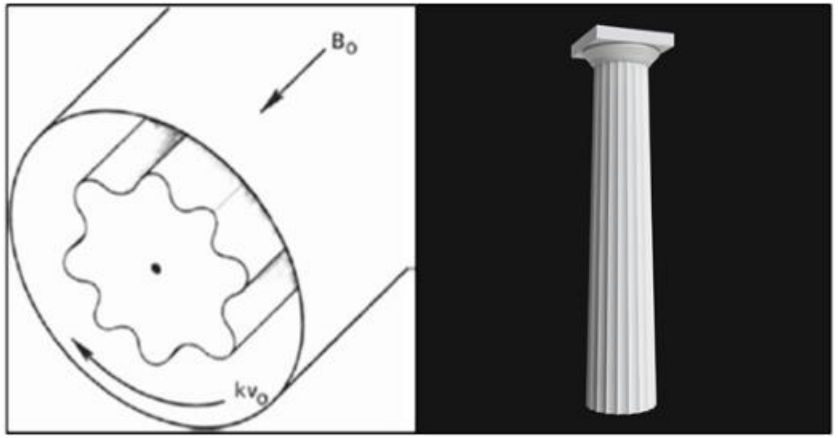
$$\begin{cases} \nabla_\parallel J_\parallel - \partial_t \nabla_\perp^2 \tilde{\varphi} + g\partial_y \tilde{\rho} = 0 \\ E_\parallel = -\partial_t \tilde{A}_\parallel - \nabla_\parallel \tilde{\varphi} = 0 \end{cases} \qquad \begin{matrix} J_\parallel = \nabla_\perp^2 A_\parallel \\ \partial_t \tilde{\rho} + \tilde{v}_r \partial_r \langle \rho \rangle = 0 \end{matrix}$$

$$\rightarrow \quad \gamma_k^2 \nabla_\perp^2 \tilde{\varphi} = -v_A^2 \nabla_\perp^2 k_\parallel^2 \tilde{\varphi} - g k_y^2 \tilde{\varphi} \partial_r \ln\langle \rho \rangle$$

$$\rightarrow \quad \boxed{\gamma_k^2 = g k_y^2 / L_\rho k_\perp^2} \quad \rightarrow$$ (favor small k_x, large k_y)



efficient for radial transport



Interchange (flute) instability

Effects of magnetic shear on instability

- What happens when magnetic shear is present?
  - Modes want to align with field lines
  - Field lines are twisted in space

When deviate from resonant surface, magnetic pitch and mode pitch **no longer match** ⇒ $k_\parallel \neq 0$.

$$k_\parallel = \boldsymbol{k} \cdot \widehat{\boldsymbol{b}} = (m/q(r) - n)/R \Big|_{r_{m,n}} = -mq'(r)x/Rq^2 = -m\hat{s}x/Rq = k_y x/L_s$$

→ $k_\parallel = 0$ at resonant surfaces

$\hat{s} = rq'/q, 1/L_s \simeq -\hat{s}/Rq, x = r - r_{m,n}$

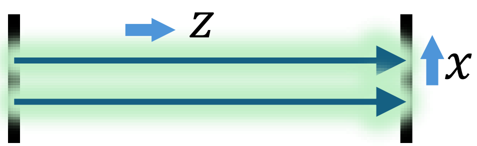
$\boldsymbol{k_\parallel = k_\theta x/L_s}$ **increases with** $\boldsymbol{x}$

- $k_\parallel \neq 0$ ⇒ field line bending ⇒ energetically unfavorable ⇒ mode stabilization

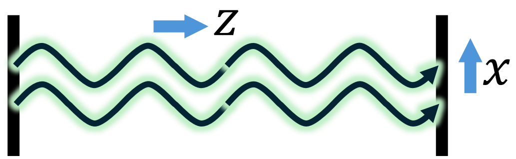
→ Suydam criterion: $-(8\pi p'/r)(q/B_z q')^2|_{r_{m,n}} > 0.25$ (critical pressure gradient)

- The farther away from the resonance surface, the stronger the line bending effect

→ mode localization



without shear



with shear

Breakdown of the frozen-in law

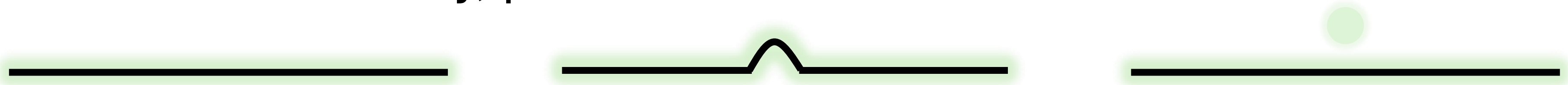
- Magnetic shear+ flux frozen-in ⇒ field line bending ⇒ stabilization

flux frozen-in law: magnetic flux through any given moving plasma element does not change.

Kelvin's theorem: vorticity through any given moving fluid element does not change.

Kelvin's theorem broken by **viscosity** ⇒ Flux frozen-in law broken by **resistivity**

- With finite resistivity, plasma can detach from field lines.



$$\partial_t \boldsymbol{B} = \underbrace{\boldsymbol{B_0}\cdot\nabla\boldsymbol{v}}_{(\boldsymbol{a})} - \boldsymbol{v}\cdot\nabla\boldsymbol{B_0} + \underbrace{\eta\nabla^2\boldsymbol{B}/4\pi}_{(b)} \quad (a)/(b) \ll 1 \text{ near resonant surfaces as } k_\parallel \to 0$$

Detachment is most notable near resonant surfaces.

- Resistivity makes interchange mode get access to free energy again!

Resistive interchange mode

- From ideal MHD to resistive MHD ⇒

Ohm's law: $-\partial_t \tilde{A}_\parallel - \nabla_\parallel \tilde{\varphi} = 0 \text{ (ideal)} \rightarrow -\nabla_\parallel \tilde{\varphi} = \eta J_\parallel \text{ (resistive)}$

$$\gamma_k^2 \nabla_\perp^2 \tilde{\varphi} - \frac{S\gamma_k}{\tau_A} \frac{k_\theta^2 x^2}{L_s^2} \tilde{\varphi} + \frac{g}{|L_\rho|} k_y^2 \tilde{\varphi} = 0 \qquad S = \tau_R / \tau_A \sim 10^7 \sim 10^9$$

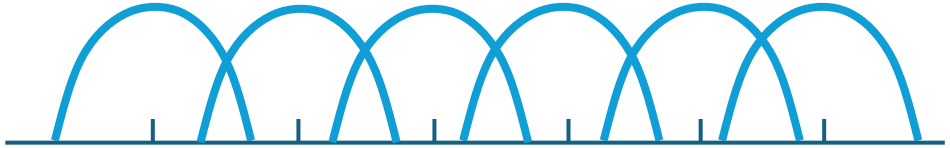
→ Harmonic oscillator!

- For resistive interchange mode:

$$\gamma \sim \mathcal{O}(\eta^{1/3}) \rightarrow \mathcal{O}(S^{-1/3}), \qquad w \sim \mathcal{O}(a/S^{1/3}) \ll a$$

localized at resonant surfaces

- Even a small resistivity can result in non-trivial effects!

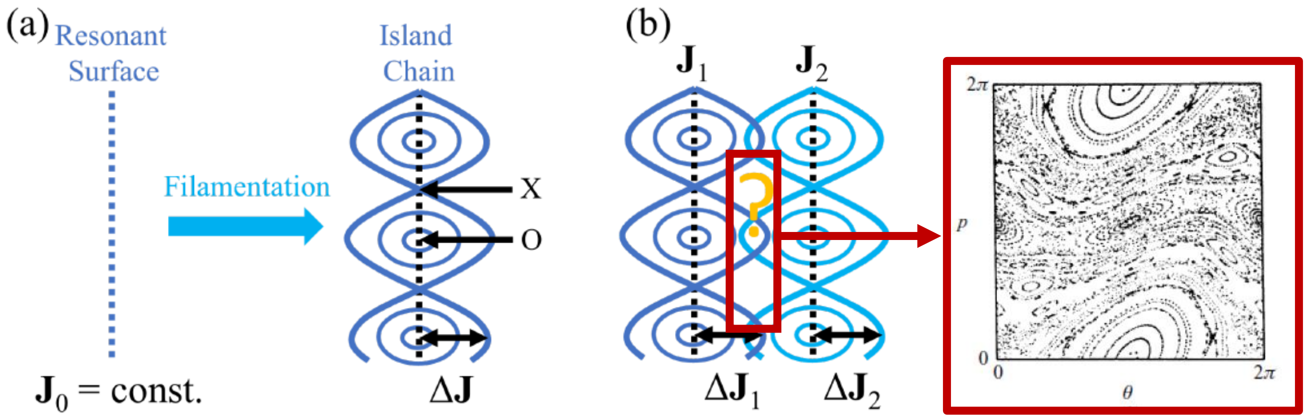


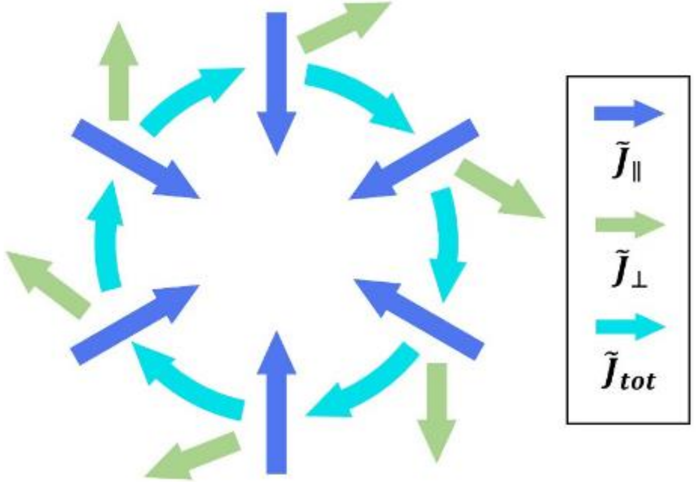
→ Modes are '**pinned**' at resonant surfaces.

- These localized modes could be an alternative mechanism account for avalanche.

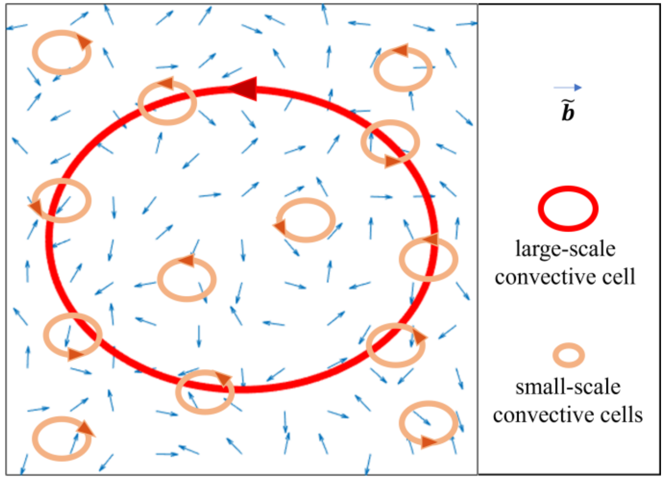
Ad 1: Resistive interchange mode in a stochastic magnetic field

Cao, M. and Diamond, P.H., 2022. PPCF.



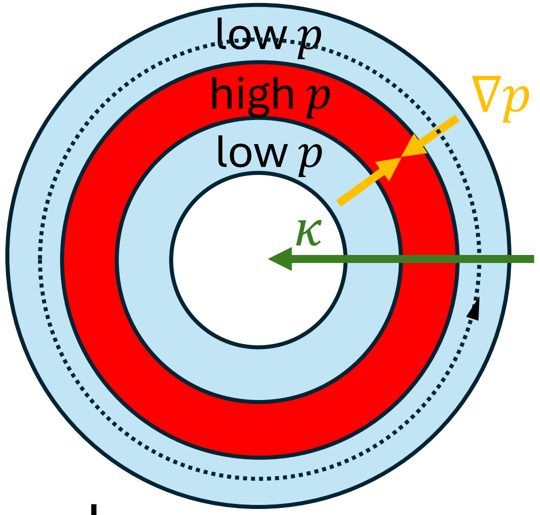


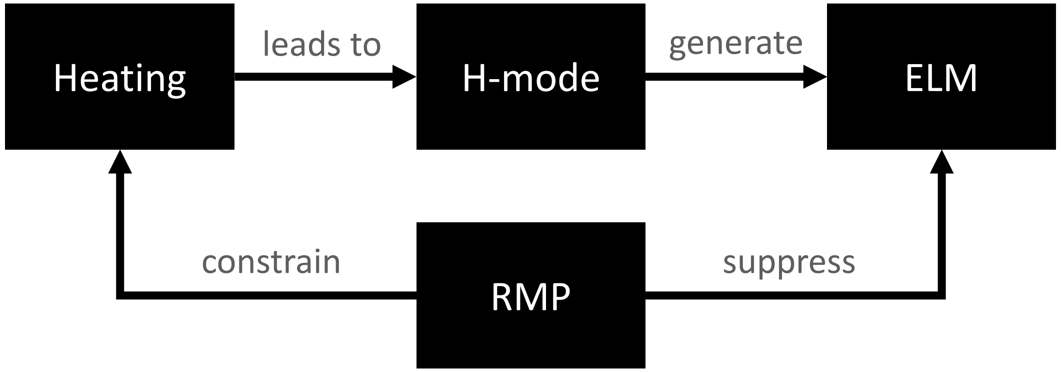
- Regular magnetic field → stochastic magnetic field
- How does stochastic magnetic field modify the dynamics of a single interchange mode?
  1. To maintain quasi-neutrality, small-scale convective cells (microturbulence) are driven.
  2. There is a non-zero $\langle \tilde{v}_r \tilde{b}_r \rangle$ correlation.
  3. Stochastic magnetic field can enhance effective plasma inertia and thus slow down mode growth.



Quasi-mode: motivation

- Resistivity restores instability, but modes are narrow → not effective for mixing.
- Introduction of quasi-mode, two purposes:
    - Broad mode structure → enhanced mixing
    - Connecting to ballooning mode, which resides in toroidal geometry
- Ballooning is very important.





Heating

leads to

H-mode

generate

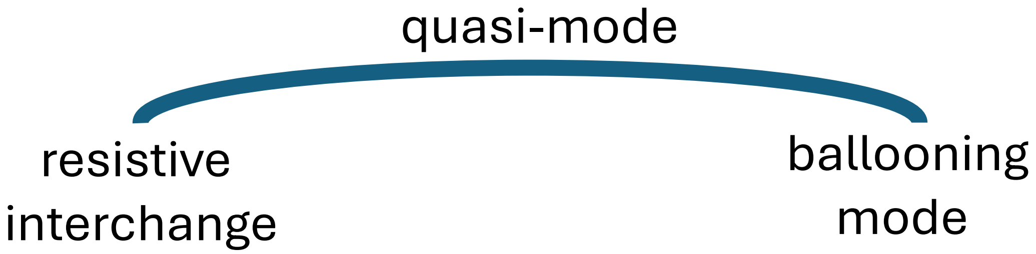
ELM

constrain

RMP

suppress

But a direct study is hard...



quasi-mode

resistive interchange

ballooning mode

Quasi-mode: basics

- Nature of quasi-mode: a wave-packet of radially localized interchange-mode at different resonant surfaces.

K.V. Roberts, J.B. Taylor, 1965. PoF.

- Wave packet ⇒

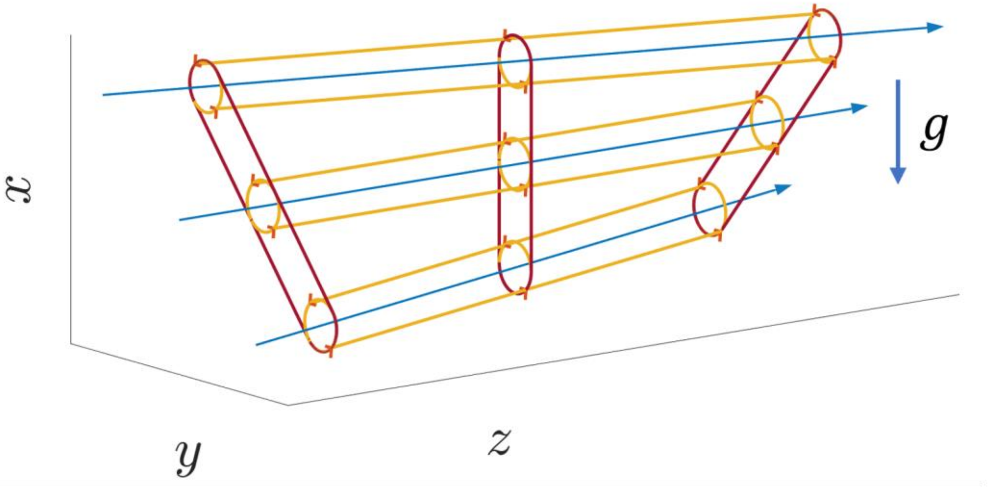
  not an eigenmode

- Dispersion?

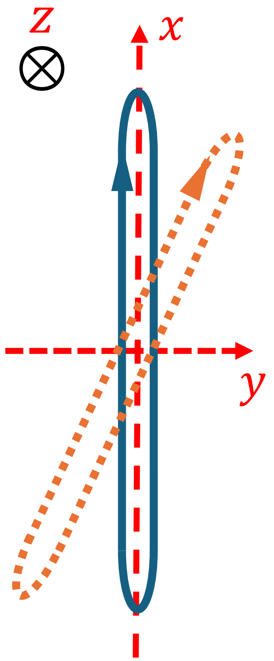
  long life-time, can maintain its shape before NL phase

- Convective rolls get twisted as moving along z to keep aligned with local field

aka: twisted slicing mode



Sketch of the quasi-mode



Quasi-mode: setup

- Basic setup:
  - Incompressible plasma
  - Finite resistivity η → resistive MHD.
  - Weak but finite magnetic shear $\boldsymbol{B_0} = (0, sx, 1)B_0$. $(s \leftrightarrow 1/L_s, sx \ll 1)$
  - Unstably stratified, $\partial_x \rho_0 = \alpha \rho_0$ $(\alpha > 0)$ with downward effective gravity.
  - Bounded in x by conducting rigid walls at $x = \pm H$; unbounded in y and z.
  - Low β → electrostatic limit.
- Model equations:

$$\rho_0 \partial_t \boldsymbol{v} = -\nabla p + (\nabla \times \boldsymbol{B}) \times \boldsymbol{B}_0/4\pi + (\nabla \times \boldsymbol{B}_0) \times B/4\pi + \rho g \qquad \nabla \cdot \boldsymbol{B} = 0 \qquad \nabla \cdot \boldsymbol{v} = 0$$

$$\partial_t \boldsymbol{B} = \nabla \times (\boldsymbol{v} \times \boldsymbol{B}_0) + \eta \nabla^2 B/4\pi \qquad \partial_t \rho = -(\boldsymbol{v} \cdot \nabla)\rho_0 = -v_x \alpha \rho_0$$

How to simplify $\boldsymbol{B} \cdot \nabla$?

$$\xrightarrow{\text{combined}} \quad \rho_0 \eta \frac{\partial^2 \nabla^2 v_x}{\partial t^2} + (\boldsymbol{B} \cdot \nabla)^2 \frac{\partial v_x}{\partial t} - \alpha g \rho_0 \eta \left(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) v_x = 0$$

Twisted slicing coordinates

- For magnetic shear

$$\hat{\boldsymbol{b}}\cdot\nabla = \nabla_\parallel = \frac{\partial}{\partial z} + \frac{1}{Rq(r)}\frac{\partial}{\partial\theta}$$

take expansion at r_0

$$\approx \frac{\partial}{\partial z} + \left(\frac{r_0}{Rq(r_0)} - \frac{rq'}{Rq^2}\bigg|_{r_0} x\right)\frac{\partial}{\partial y}$$

define $\hat{s} = \frac{rq'}{q}$

$$= \frac{\partial}{\partial z} + \left(\frac{r_0}{Rq(r_0)} - \frac{\hat{s}}{Rq}\bigg|_{r_0} x\right)\frac{\partial}{\partial y} = \frac{\partial}{\partial z} + \left(\cancel{\frac{r_0}{Rq(r_0)}} + \frac{x}{L_s}\right)\frac{\partial}{\partial y}$$

define $1/L_s \simeq -\hat{s}/Rq$

→ absorb to ∂_z

$$\begin{aligned} x' &= x \\ y' &= y - xz/L_s - r_0 z/Rq(r_0) \\ z' &= z \\ t' &= t \end{aligned}$$

→

$$\begin{aligned} \partial_x &= \partial_{x'} - z'\partial_{y'}/L_s \\ \partial_y &= \partial_{y'} \\ \partial_z &= \partial_{z'} - x'\partial_{y'}/L_s - r_0\partial_{y'}/Rq(r_0) \\ \partial_t &= \partial_{t'} \end{aligned}$$

→

$$\nabla_\parallel = \partial_{z'}$$

shearing term removed!

Quasi-mode: model equation

- Transform into twisted slicing coordinates

$$\xi = x, \qquad \chi = y - sxz, \qquad \zeta = z \qquad\qquad (\zeta\text{: length along the main field line})$$

- Look for solutions of the form (not periodic in z):

$$v_x = v(\zeta)\exp(ik_x\xi + ik_y\chi)\exp pt$$

- Then we get:

$$(1 + \cancel{\epsilon^2 q})\frac{1}{k_y^2}\frac{\partial^2 v}{\partial\zeta^2} - \cancel{2\epsilon^2 qis\xi\frac{1}{k_y}\frac{\partial v}{\partial\zeta}} - \epsilon^2\left[q(1 + \cancel{s^2\xi^2}) + \frac{p^2}{\alpha g}s^2\zeta^2 - \cancel{\frac{p^2}{\alpha g}\left(\frac{k_x^2}{k_y^2} - 2s\zeta\frac{k_x}{k_y}\right)}\right]v = 0$$

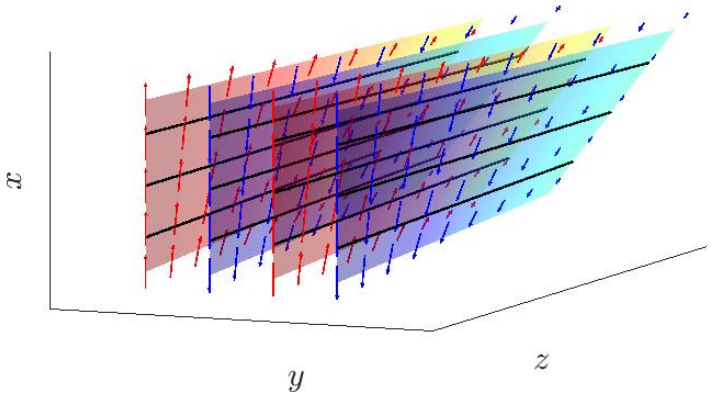
$$\epsilon^2 = \frac{\alpha g\rho_0\eta}{pB_0^2} \qquad q = \frac{p^2}{\alpha g} - 1 \xrightarrow[\epsilon \sim (k_y\Delta)^{-1} \ll 1 \text{ (long mode length)}]{k_x/k_y \ll 1 \text{ (broad mode structure)} \quad s\xi \ll 1 \text{ (weak shear)}}$$

Equation for quasi-mode:
$$\frac{d^2v}{d\zeta^2} - \frac{p\rho_0\eta}{B_0^2}(sk_y)^2\zeta^2 v + \frac{p\rho_0\eta k_y^2}{B_0^2}\left(\frac{\alpha g}{p^2} - 1\right)v = 0$$

Quasi-mode: solutions

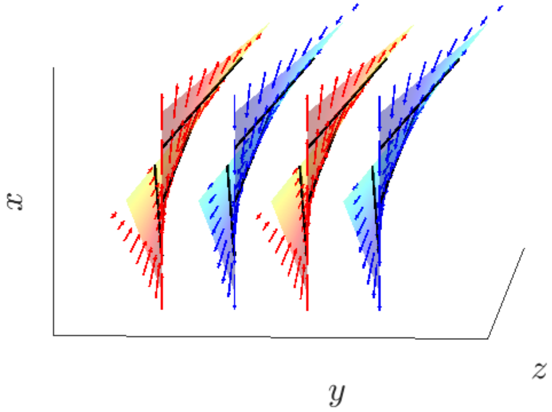
- Another harmonic oscillator?

$$\frac{d^2v}{d\zeta^2} - \frac{p\rho_0\eta}{B_0^2}\left(sk_y\right)^2\zeta^2 v + \frac{p\rho_0\eta k_y^2}{B_0^2}\left(\frac{\alpha g}{p^2} - 1\right)v = 0$$



- **Not a coincidence.**

$\boxed{\zeta \leftrightarrow k_\parallel \propto x}$ H.O. for x → H.O. for z



- Solution:

broad structure ⇒ efficient for mixing

$$v_x(x, y, z) = \underline{g(x)}v_j(z)\exp\left[ik_y(y - sxz)\right]$$

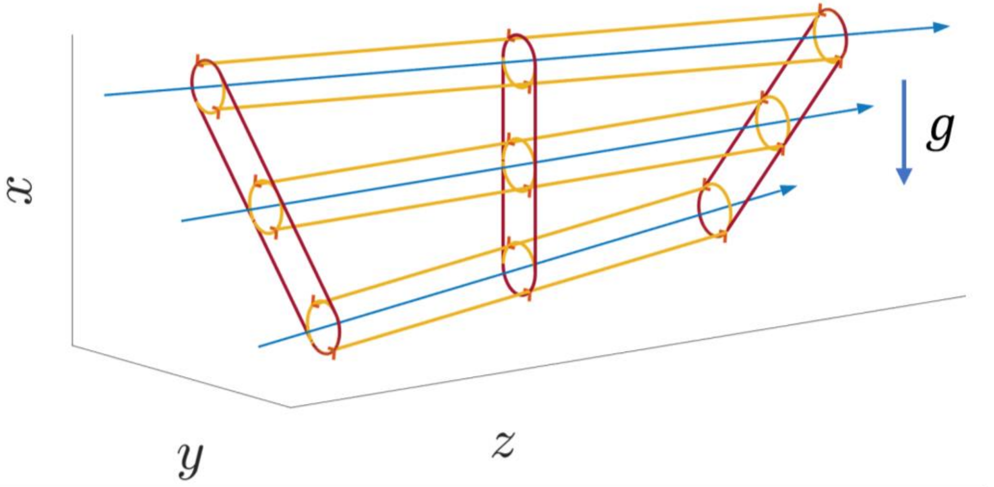
$$v_j(\zeta) = 2^{-j/2}\exp(-\xi^2/2\,\Delta^2)H_n(\zeta/\Delta)$$

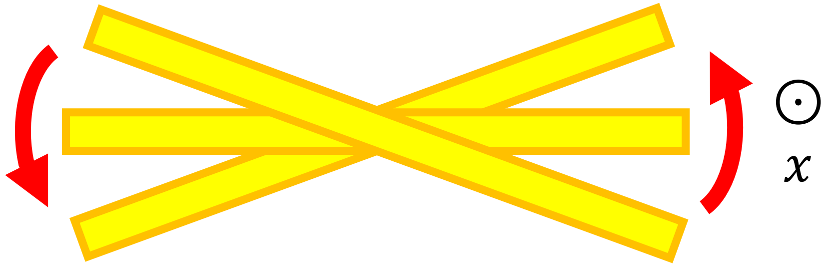
$$p_j = (\alpha g)^{2/3}\left(\frac{\tau_A k_y^2}{Ss^2}\right)^{1/3}(2j+1)^{-2/3} \qquad \Delta_j = \frac{1}{(\alpha g)^{1/6}}\left(\frac{S}{\tau_A k_y^2 s}\right)^{1/3}(2j+1)^{1/6} \propto \eta^{-1/3}$$

Quasi-mode: physical interpretation

- Quasi-mode has finite length in the main field direction

  ⟷ resistive interchange mode
- Physics:
  - Magnetic shear → rotation of plasma filaments when moving up and down.
  - Perfect alignment with field ling → infinite length in the z direction → divergent rotational kinetic energy
  - Adjust to a finite mode length Δ → plasma detached from field lines → enhanced resistive dissipation
- Balance among E_g, E_{tot}, and E_{diss}





top view of the rotation of filaments

Quasi-mode: relation to resistive interchange

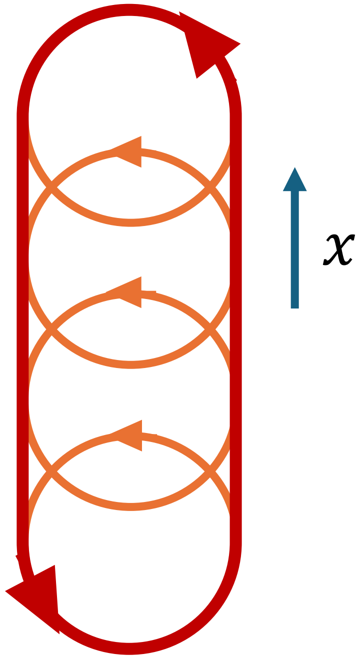
- Can show explicitly that quasi-mode is a linear superposition of resistive interchange mode.

- Resistive interchange:

$$v_x = v_g(x)\exp\left(\gamma_{\boldsymbol{k}}t + ik_y y + ik_z z\right)$$

$$v_g^j(X) = e^{-j/2}H_j((x-x_0)/\delta_{\boldsymbol{k}})\exp\left(-(x-x_0)^2/2\delta_{\boldsymbol{k}}^2\right)$$

$$\gamma_{\boldsymbol{k}}^j = (\alpha g)^{\frac{2}{3}}\left(\frac{\tau_A k^4}{Ss^2k_y^2}\right)^{\frac{1}{3}}(2j+1)^{-\frac{2}{3}} \quad x_0 = -\frac{k_z}{sk_y} \quad \delta_{\boldsymbol{k}} = (\alpha g)^{\frac{1}{6}}\left(\frac{\tau_A k}{Ss^2k_y^2}\right)^{\frac{1}{3}} \propto \eta^{\frac{1}{3}}$$



- Summing over modes with same k_y at different resonant surfaces:

$$u = \exp ik_y y\int f(k_z)\exp\left[ik_z z - X^2/2\delta_{\boldsymbol{k}}^2\right]\exp\gamma_{\boldsymbol{k}}t\,dk_z \qquad 1/\Delta \sim sk_y\delta_{\boldsymbol{k}} \sim k_\parallel$$

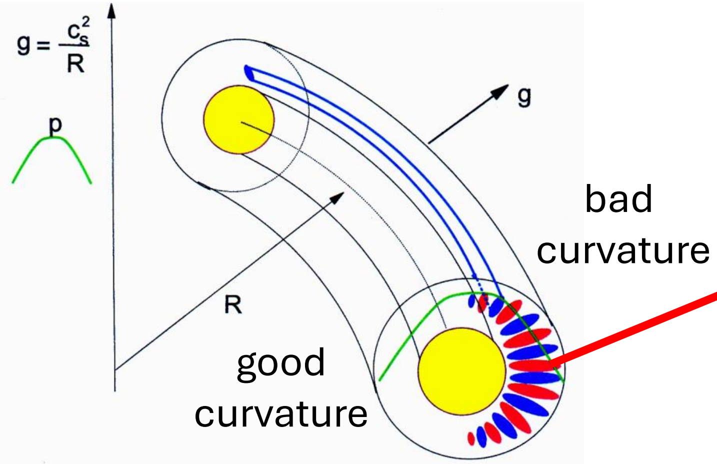
$$\cong \delta\sqrt{2}\pi g(x)\exp\left[ik_y(y-sxz) - \frac{\left(sk_y\delta_0 z\right)^2}{2} + \gamma t\right] \rightarrow \text{quasi-mode}$$

Ad 2: Quasi-mode evolution in a stochastic magnetic field

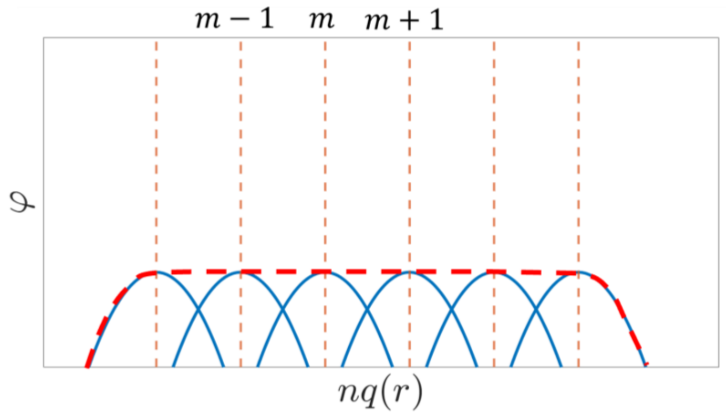
Cao, M. and Diamond, P.H., 2024. NF.

- How does stochastic magnetic field affect quasi-mode?
  - Quasi-mode is a wave-packet of resistive interchange ⇒ similar results expected
  - But mode structures are quite different ⇒ something should change
- Results:
  - Appearance of small-scale convective cells.
  - Stabilization of quasi-mode via enhanced inertia. Stronger for quasi-mode.
  - Turbulent damping arising from microturbulence. Stronger for quasi-mode.
  - No-trivial correlation $\langle \tilde{b}_r \tilde{v}_x \rangle$. $\langle \tilde{b}_y \tilde{v}_x \rangle$ is also non-zero.
- **All the changes can be attributed to the change in mode structure and spatial ordering.**

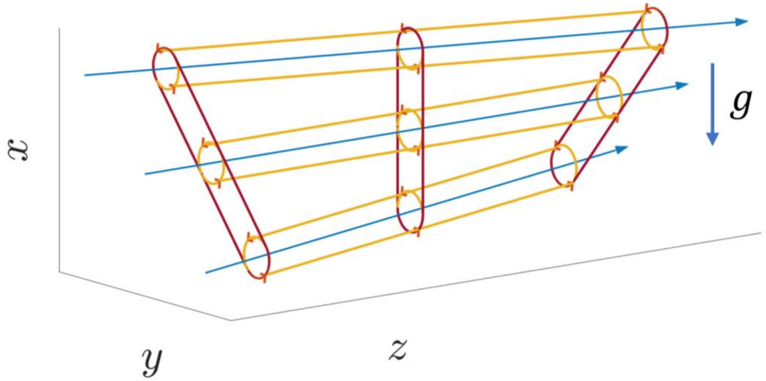
Welcome to the world of the torus



No poloidal symmetry



coupling of poloidal harmonics due to toroidicity



superposition of resistive interchange modes

ballooning

- physical processes driven by curvature in the “bad curvature” region
- modes/fluctuations where toroidicity effect matters

- A quasi-mode in a cylinder resembles a ballooning mode in a torus
- Approaches: **Bloch eigenmode equation**; **ballooning mode representation**

Bloch eigenmode equation

- What happens with toroidicity? → magnetic drift

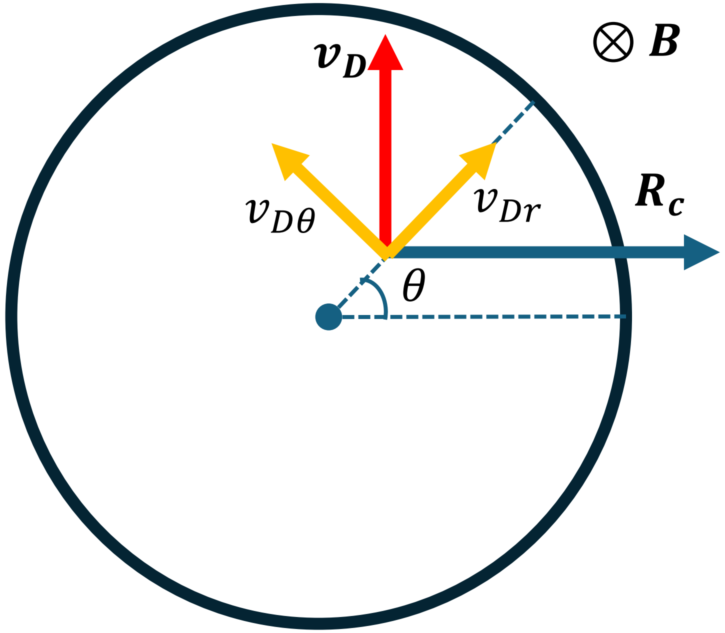
$$v_D = v_R + v_{\nabla B} = \left(v_\parallel^2 + \frac{1}{2}v_\perp^2\right)\frac{m}{q}\,\frac{\boldsymbol{R_c}\times\boldsymbol{B}}{R_c^2B^2}$$

- Ion continuity equation:

$$\partial_t\tilde{n} + \boldsymbol{v_{Di}}\cdot\nabla\tilde{n} + \tilde{v}_r\partial_r n_0 + n_0\nabla\cdot\tilde{\boldsymbol{v}} + n_0\nabla_\parallel\tilde{v}_\parallel = 0$$

$$\partial_t\tilde{v}_\parallel = -|e|\nabla_\parallel\tilde{\varphi}/m$$ + Boltzmann electron

$$\omega_{Di} = \boldsymbol{v_{Di}}\cdot\nabla = v_{Di}\left(\cos\theta\,\partial_y + \sin\theta\,\partial_x\right)$$



$$T, T' \sim \omega_{Di}/\omega \sim \epsilon_T = L_n/R$$

$$\longrightarrow \underbrace{\left[-\rho_s^2\frac{\partial^2}{\partial x^2} + k_y^2\rho_s^2 + \left(1 - \frac{\omega^*}{\omega}\right) - \frac{k_\theta^2x^2}{L_s^2\omega^2}c_s^2\right]}\varphi_m + T(\varphi_{m+1} + \varphi_{m-1}) + T'(\varphi_m - \varphi_{m-1}) = 0$$

drift acoustic $\longleftarrow$ $k_\perp^2\rho_s^2 + 1 - \omega^*/\omega - k_\parallel^2c_s^2/\omega^2 = 0$

$$T(\varphi_{m+1} + \varphi_{m-1}) = \frac{v_{Di}}{2}\partial_y(\varphi_{m+1} + \varphi_{m-1})$$

$$T'(\varphi_{m+1} - \varphi_{m-1}) = \frac{v_{Di}}{2}\partial_x(\varphi_{m+1} - \varphi_{m-1})$$

Bloch eigenmode equation

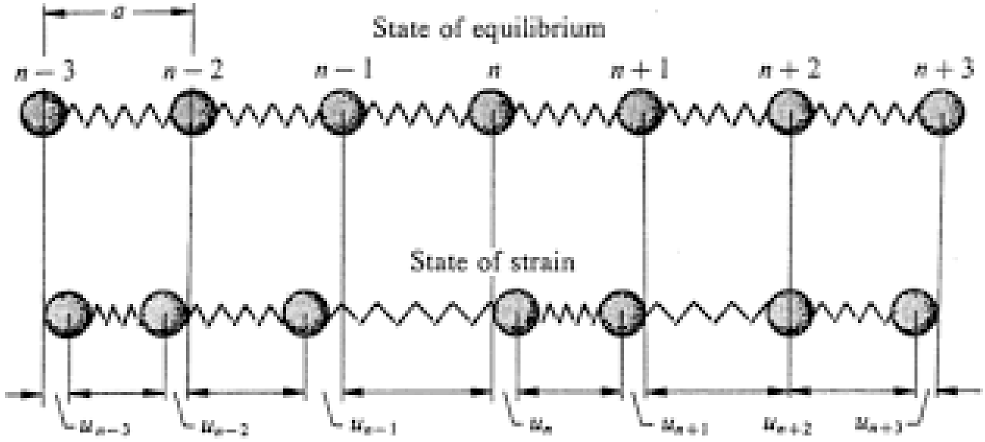
- Observations:
  - n remains as good ‘quantum’ number.
  - magnetic drift due to toroidicity effect tends to linearly couple poloidal harmonics.
- Rewrite the equation into:

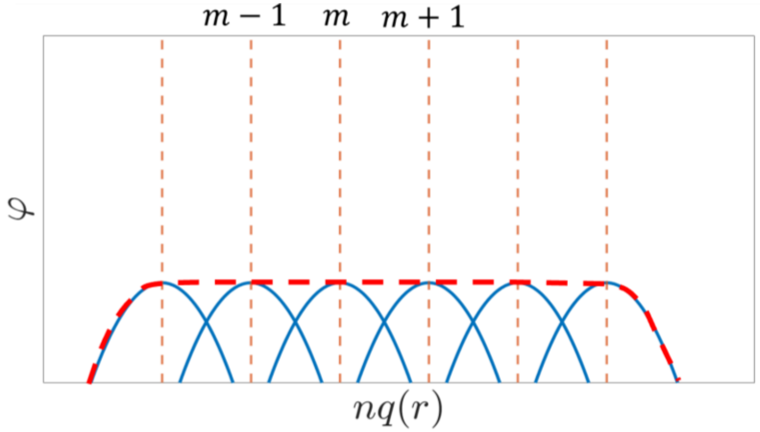
$$L_m\varphi_m + T_{m+1}\varphi_{m+1} + T'_{m-1}\varphi_{m-1} = 0$$

➡ equivalent to a tri-diagonal matrix equation

$$\begin{bmatrix} \cdots & \cdots & 0 & 0 & 0 & 0 \\ T'_{m-2} & L_{m-1} & T_m & 0 & 0 & 0 \\ 0 & T'_{m-1} & L_m & T_{m+1} & 0 & 0 \\ 0 & 0 & T'_m & L_{m+1} & T_{m+2} & 0 \\ 0 & 0 & 0 & T'_{m+1} & L_{m+2} & T_{m+3} \\ 0 & 0 & 0 & 0 & \cdots & \cdots \end{bmatrix} \begin{bmatrix} \cdots \\ \varphi_{m-1} \\ \varphi_m \\ \varphi_{m+1} \\ \varphi_{m+2} \\ \cdots \end{bmatrix} = 0$$

1D linear harmonic oscillator chain





Bloch eigenmode equation

- Problem reduces to Bloch eigenmode problem.
- Toroidicity renders problem equivalent to linear chain.
- Toroidicity introduces a new scale→ Δr: spacing between two adjacent harmonics with same n

$$\frac{q\left(r_{m+1,n}\right) - q\left(r_{m,n}\right)}{\Delta r} = \left(\frac{m+1}{n} - \frac{m}{n}\right)/\Delta r = \frac{1}{n\Delta r} = q' \Rightarrow \Delta r = \frac{1}{nq'} = \frac{r}{nq\hat{s}} = \frac{1}{k_y\hat{s}}$$

- n large, k_y large, then Δr small

→ adjacent harmonics have same ‘shape’ (translational invariance)

$$\varphi_{m-1} = \varphi_m(x + \Delta r), \qquad \varphi_{m+1} = \varphi_m(x - \Delta r)$$

$$L_m \varphi_m(x) + T_{m+1}\varphi_m(x - \Delta) + T'_{m-1}\varphi_m(x + \Delta) = 0$$

Bloch eigenmode equation

- Fourier transform (with respect to x):

$$\longrightarrow \left[\rho_s^2 \left(k_y s\right)^2 \hat{\eta}^2 + \frac{c_s^2}{\omega^2 R^2 q^2} \frac{d^2}{d\hat{\eta}^2} + \left(1 - \frac{\omega^*}{\omega}\right) + \rho_s^2 k_y^2\right] \hat{\varphi}_m + \frac{v_b}{\omega} k_y (\cos\hat{\eta} + s\hat{\eta}\sin\hat{\eta})\hat{\varphi}_m = 0$$

- Recall $k_\parallel \propto x$, then η is again the distance along the main field line.

→ It determines how mode varies along field line → mode structure.

- Δr^{-1} vs. φ_m'/φ_m, i.e., spacing compared to mode width.
  - If $\Delta r \varphi_m'/\varphi_m < 1 \Rightarrow$ adjacent harmonics strongly overlap $\Rightarrow$ "strong" Ballooning
  - If $\Delta r \varphi_m'/\varphi_m > 1 \Rightarrow$ "weak" ballooning

↓

expand $\varphi_m(x + \Delta r)$

Ballooning mode representation

- Ballooning mode: $k_\perp \gg k_\parallel$. In slowly varying medium, the eikonal form is:

$$\varphi = F(\psi,\chi)\exp\left[in\left(\zeta - \int^{\chi} \nu d\chi'\right)\right]$$

$\psi \leftrightarrow x$: magnetic flux coordinate
$\chi \leftrightarrow \theta$: poloidal angle-like variable
ζ: toroidal angle

F is a slowly varying function. $\nu(\psi,\chi) = d\zeta/d\chi$ $\quad q = \oint \nu d\chi/2\pi$ $\quad n \gg 1$

- No poloidal symmetry → m is no longer a good 'quantum' number

Connor, J.W., Hastie, R.J. and Taylor, J.B., 1979.

- **Warning**: still have poloidal periodicity → $\varphi(\chi = 0) = \varphi(\chi = 2\pi)$

$$F(\psi_1,\chi = 0) = F(\psi_1,\chi = 2\pi)\exp\left[-in\int^{2\pi} \nu(\chi',\psi_1)d\chi'\right]$$

varies a little ↓ varies a little ↓ varies a lot ↓

$$F(\psi_2,\chi = 0) = F(\psi_2,\chi = 2\pi)\exp\left[-in\int^{2\pi} \nu(\chi',\psi_2)d\chi'\right]$$

$$\psi_2 = \psi_1 + d\psi$$

- Reconcile $k_\perp/k_\parallel$ with periodicity in a sheared magnetic field: **ballooning mode representation.**

Ballooning mode representation

Not restricted to ballooning mode!

- Basic idea: if $\hat{\varphi}(\eta, x)$ is a solution of a 2D eigenvalue problem

$$L(\eta, x)\hat{\varphi}(\eta, x) = \lambda\hat{\varphi}(\eta, x), \qquad \eta \in (-\infty, \infty)$$

then

$$\varphi(\theta, x) = \sum_m e^{-im\theta} \int_{-\infty}^{\infty} e^{im\eta}\hat{\varphi}(\eta, x)d\eta\,, \qquad \theta \in (0,2\pi]$$

is a periodic eigen solution for periodic operator $L(\theta = 0) = L(\theta = 2\pi)$, i.e.,

$$L(\theta, x)\varphi(\theta, x) = \lambda\varphi(\eta, x).$$

Proof: for simplicity and without loss of generality, suppose $L(\theta, x) = \partial/\partial\theta$

$$(\partial_\theta - \lambda)\varphi(\theta, x) = \sum_m (-im - \lambda)e^{-im\theta} \int_{-\infty}^{\infty} e^{im\eta}\hat{\varphi}(\eta, x)d\eta$$

$$= \sum_m e^{-im\theta} \int_{-\infty}^{\infty} (-im - \lambda)e^{im\eta}\hat{\varphi}(\eta, x)d\eta = \sum_m e^{-im\theta} \int_{-\infty}^{\infty} e^{im\eta}(\partial_\eta - \lambda)\hat{\varphi}(\eta, x)d\eta$$

- Poloidal periodicity is relaxed for $\hat{\varphi}(\eta, x) \Rightarrow$ safe to use eikonal form for $\hat{\varphi}(\eta, x)$.

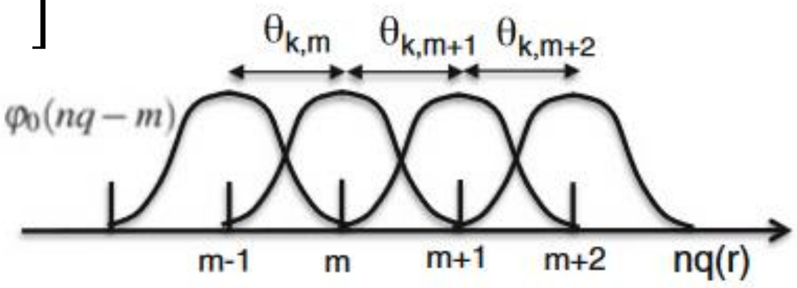
Ballooning mode representation

- The eikonal form of $\hat{\varphi}(\eta, x)$:

$$\hat{\varphi}(\eta, x, \zeta) = F(\eta, x)\exp\left[in\left(\zeta - \int_{\eta_0}^{\eta} \nu d\eta'\right)\right] = \varphi_0(\eta, x)\exp\left[in\left(\zeta - q\eta + \cancel{\int_0^q \theta_k dq}\right)\right]$$

$$\longrightarrow \quad \varphi(\theta, x, \zeta) = \sum_m e^{i(n\zeta - m\theta)} \int_{-\infty}^{\infty} \varphi_0(\eta, x)\exp\left[i\,\underbrace{(m - nq)}_{k_\parallel}\,\eta\right] d\eta$$

$\longrightarrow$ η is distance along field line.



- Recall quasi-mode $v_x = g(x)v_j(z)\exp\left[ik_y(y - sxz)\right]$

$$\varphi(\theta, x) = \int_{-\infty}^{\infty}\sum_m \exp[-im(\theta - \eta)]\hat{\varphi}(\eta, x)d\eta = \sum_N \hat{\varphi}(\theta - 2\pi N, x) \longleftarrow$$

superposition of quasi-modes

$$\varphi(\theta, x) = \sum_m e^{-im\theta}\int_{-\infty}^{\infty} \varphi_0(\eta, x)\exp[i(m - nq)\eta]\, d\eta = \sum_m e^{-im\theta}\, \boxed{\varphi_m(m - nq(x), x)}$$

poloidal harmonics

Beyond linear theory

Magnetic shear enhanced decorrelation

- For the sheared magnetic field in tokamak:

$\boldsymbol{B}\cdot\nabla\psi=0$ ψ is magnetic density flux $\tilde{b}_r$: stochastic magnetic field

$$\Rightarrow\frac{\partial\psi}{\partial z}+\frac{B_\theta}{B_0}\frac{\partial\psi}{\partial y}+\tilde{b}_r\frac{\partial\psi}{\partial r}=0\quad\psi=\bar{\psi}+\tilde{\psi}$$

magnetic diffusivity

$$D_M=\sum_k\left|\tilde{b}_{rk}\right|^2\pi\delta(k_\parallel)$$

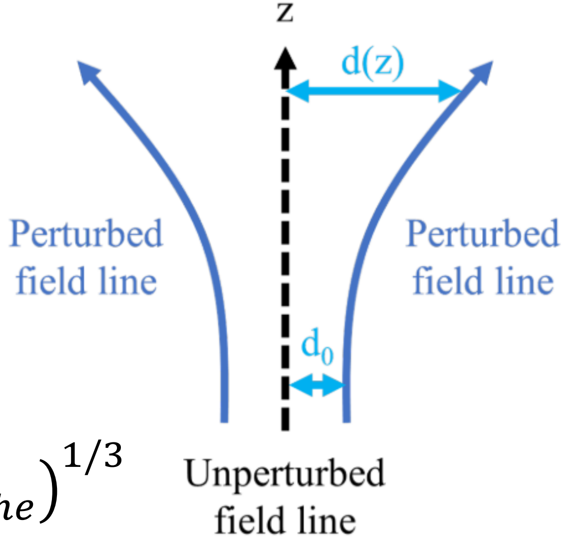
$$\Rightarrow\tilde{\psi}_k=\frac{-i}{k_z-k_\theta B_\theta/B_0}\tilde{b}_{r,k}\frac{\partial}{\partial r}\bar{\psi}\Rightarrow\frac{\partial\bar{\psi}}{\partial z}+\frac{x}{L_s}\frac{\partial\bar{\psi}}{\partial y}-D_M\frac{\partial^2\bar{\psi}}{\partial r^2}=0$$

Transform to twisted slicing coordinates:

$$\frac{\partial\bar{\psi}}{\partial z'}-D_M\left(k'_x-k'_y\frac{z'}{L_s}\right)^2\bar{\psi}=0\Rightarrow\bar{\psi}_k\propto\exp\left(-\frac{{k'_y}^2D_Mz^3}{3L_s^2}\right)$$

$l_c=\left(\frac{{k'_y}^2D_M}{3L_s^2}\right)^{1/3}$: decorrelation length in the main field direction.

For electron, can define decorrelation time as $\tau_c=\left({k'_y}^2D_M/3L_s^2v_{the}^3\right)^{1/3}$



How geometry affects nonlinear coupling

- A frequently encountered operator:

$$\tilde{\boldsymbol{b}} \cdot \nabla \tilde{v}_x = -\nabla_\perp \tilde{A} \times \hat{\boldsymbol{z}} \cdot \nabla_\perp \tilde{v}_x = \sum_{\boldsymbol{k},\boldsymbol{k}'} \tilde{A}_{\boldsymbol{k}} \tilde{v}_{x\boldsymbol{k}'} \hat{\boldsymbol{z}} \cdot (\boldsymbol{k}'_\perp \times \boldsymbol{k}_\perp) \exp[i(\boldsymbol{k}+\boldsymbol{k}') \cdot \boldsymbol{r}]$$

- Need to evaluate the coupling factor $\hat{\boldsymbol{z}} \cdot (\boldsymbol{k}'_\perp \times \boldsymbol{k}_\perp)$.

- For resistive interchange mode: $\tilde{v}_x = \sum_{\boldsymbol{k}_\perp} \tilde{v}_{x\boldsymbol{k}} \exp\left[i\left(k_x x + k_y y + i k_z z\right)\right]$

$$\hat{\boldsymbol{z}} \cdot (\boldsymbol{k}'_\perp \times \boldsymbol{k}_\perp) = \left(k'_x k_y - k'_y k_x\right) \neq 0.$$

- For quasi-mode: $\tilde{v}_{x\boldsymbol{k}} = v_n(z) \exp i k_y (y - sxz)$

$$\hat{\boldsymbol{z}} \cdot (\boldsymbol{k}'_\perp \times \boldsymbol{k}_\perp) = -k'_y k_y sz + k'_y k_y sz = 0.$$

→ The coupling factor between two wave-packets is 0.

How geometry affects nonlinear coupling

Saturation mechanism of a mode?

- Transfer to other mode ⇒ dissipation
- Use up free energy ⇒ feedback on the mean profile

→ No energy transfer between modes

→ Can only feed back on the mean gradient.

→ Relax to near marginality

- $\langle n \rangle$: plateau formation?
- $\langle \varphi \rangle$: zonal mode?

How geometry affects nonlinear coupling

Frieman, E.A. and Chen, L., 1981. PoF.

- For ballooning mode:

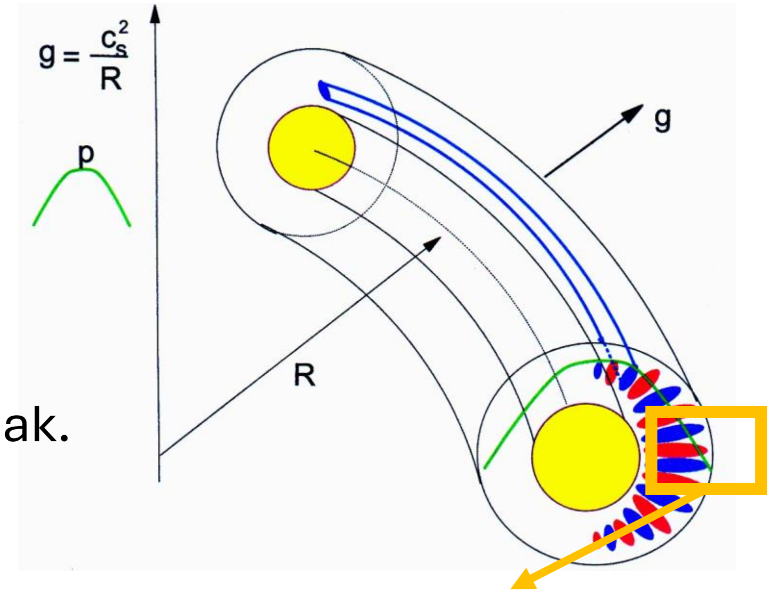
$$\tilde{v}_x(\theta,x,\phi)=\sum_{m,n} e^{i(n\phi-m\theta)}\int_{-\infty}^{\infty} v_0(\eta,x)\exp\left[i\left(m\eta_n-n\int^{\eta_n}\nu d\eta'\right)\right]d\eta_n$$

- The coupling factor becomes

$$c=\hat{\boldsymbol{z}}\cdot(\boldsymbol{k}'_\perp\times\boldsymbol{k}_\perp)\propto nn'\int_{\eta_n}^{\eta_{n'}}\frac{\partial\nu}{\partial x}d\eta$$

η_n and $\eta_{n'}$ are coordinates along the main field line of two different modes.

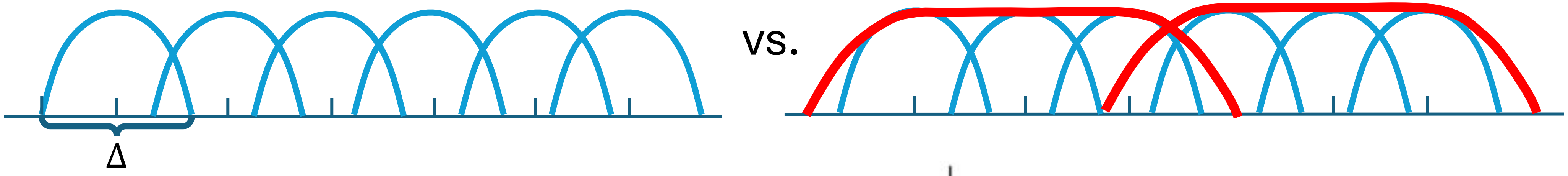
- Ballooning is strongest near outer mid-plane. If modes are concentrated in the outer mid-plane, coupling would be weak.
  - ⇒ feedback on the gradient (mean field evolution)
  - ⇒ competition between stability and nonlinear coupling.



ballooning strongest

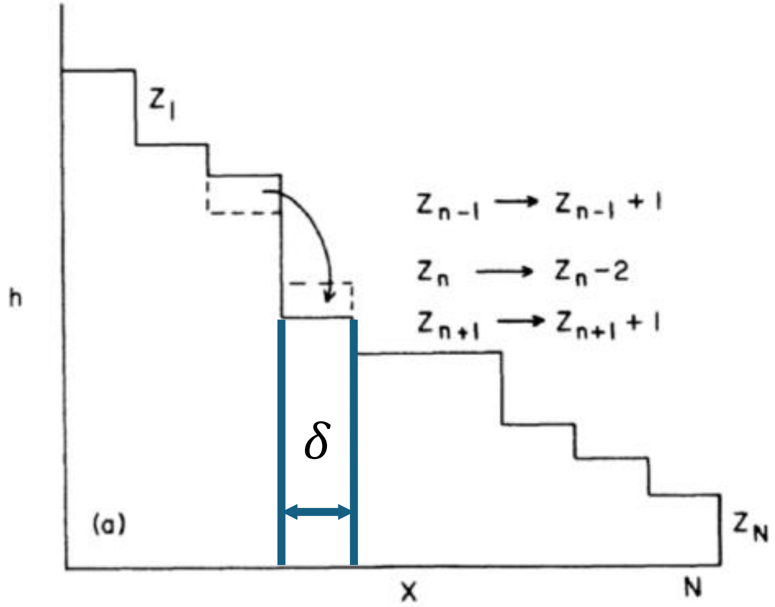
Indication for avalanche

- Mode overlapping ⇒ avalanche



- Two ways to have avalanche:
  1. Coupling of Localized modes
     - resistive interchange → wave-packet
     - poloidal harmonics → ballooning
     - Similarity to sand-pile model: unit size much smaller than system size

       $\rho_* \sim \Delta/a \ll 1, \qquad \delta/L \ll 1$
  2. Interactions of ballooning mode
- Which one is the criminal remains unclear…



sand pile model

Summary

- We explain how geometry affects turbulence, instability, and transport.
- In periodic cylinder:
  - Field pitch = mode pitch → resonant surfaces → habitat for instability
  - Magnetic shear → mode stabilization and localization + **enhanced decorrelation**
  - Resistivity → detachment of fluids from fields → restoration of instability
  - Wave-packet → broader structure → **enhanced mixing** & **reduced mode coupling**
- In torus:

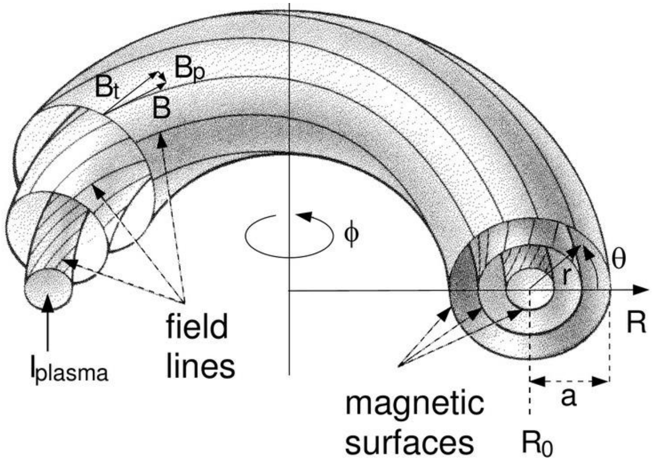
an alternative picture for avalanche

↑

  - Toroidicity → coupling of poloidal harmonics → ballooning mode
  - Bloch eigenmode equation & ballooning mode representation
  - **Stability vs. nonlinear coupling**

Side story: velocity shear

- Operator $\boldsymbol{B}\cdot\nabla$ is important!
- Similar in structure to velocity shear
- Rewrite $\boldsymbol{B}\cdot\nabla$ in detail:



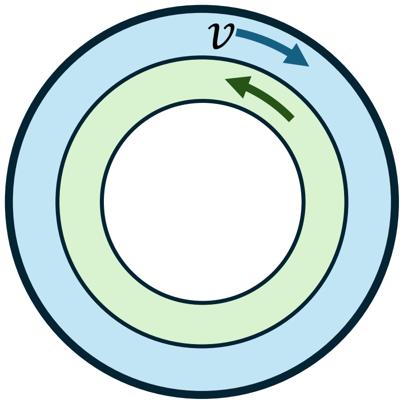
$$\hat{\boldsymbol{b}}\cdot\nabla=\frac{\partial}{R\partial\phi}+\frac{B_\theta(r)}{B_0 r}\frac{\partial}{\partial\theta}+\tilde{\boldsymbol{b}}\cdot\nabla$$

$$=\partial_z+\underline{\partial_\theta/Rq(r)}+\tilde{\boldsymbol{b}}\cdot\nabla$$

- $z\leftrightarrow t, r\leftrightarrow x,\ rd\theta\leftrightarrow dy\Rightarrow$ analogous to:

shearing term

$$d_t=\partial_t+\underline{\bar{v}_y(x)\partial_y}+\tilde{\boldsymbol{v}}\cdot\nabla$$



- Shear flow also enters Landau resonance, i.e., $1/[\omega-k_y\langle v_y(x)\rangle]$

Side story: Velocity shear

- How to simplify $\partial_t + \bar{v}_y(x)\partial_y$?

Goldreich, P. and Lynden-Bell, D., 1965.

- Shearing coordinates (same as twisted slicing coordinates)

⇒ a natural way to describe fluctuations in shear

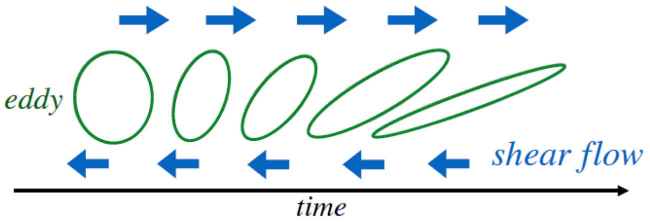
$$\begin{aligned} x' &= x \\ y' &= y - \bar{v}_y(x)t \\ z' &= z \\ t' &= t \end{aligned} \xrightarrow[\text{local expansion}]{\text{linear shear}} \begin{aligned} x' &= x \\ y' &= y - \bar{v}_y' x t \\ z' &= z \\ t' &= t \end{aligned} \longrightarrow \begin{aligned} \partial_x &= \partial_{x'} - \bar{v}_y' t' \partial_{y'} \\ \partial_y &= \partial_{y'} \\ \partial_z &= \partial_{z'} \\ \partial_t &= \partial_{t'} - \bar{v}_y x' \partial_{y'} \end{aligned}$$

$$\partial_t + \bar{v}_y' x \partial_y = \partial_{t'} - \cancel{\bar{v}_y x' \partial_{y'}} + \cancel{\bar{v}_y' x' \partial_y} = \partial_{t'} \longrightarrow$$ shearing term eliminated!

- Can connect wave numbers in shearing coordinates and usual coordinates:

$$c_{k'} \exp i\boldsymbol{k}' \cdot \boldsymbol{x}' = c_{k'} \exp i\left(\boldsymbol{k}' \cdot \boldsymbol{x} - k_y' \bar{v}_y' x t\right)$$

$$\Rightarrow \boxed{k_x = k_x' - k_y' \bar{v}_y' t}$$ eddy tilting



Side story: velocity shear

- For diffusion with the presence of linear shear flow

$$\left[\frac{\partial}{\partial t}+\bar{v}_y' x\,\frac{\partial}{\partial y}-D\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\right]c=0$$

c: passive scaler

D: diffusivity

Transform to shearing coordinates:

shear enhanced diffusion

↑

$$\left\{\frac{\partial}{\partial t'}+D\left[\left(k_x'-k_y'\bar{v}_y't'\right)^2+{k_y'}^2\right]\right\}c_{k'}=0\Rightarrow$$

$$c_{k'}=c_0\exp ik_y'y'\exp\left(-{k_y'}^2Dt\right)\exp\left[-\int^t dt'D\left(k_x'-k_y'\bar{v}_y't'\right)^2\right]\propto\exp\left[-\frac{{k_y'}^2D\bar{v}_y't^3}{3}\right]$$

↓

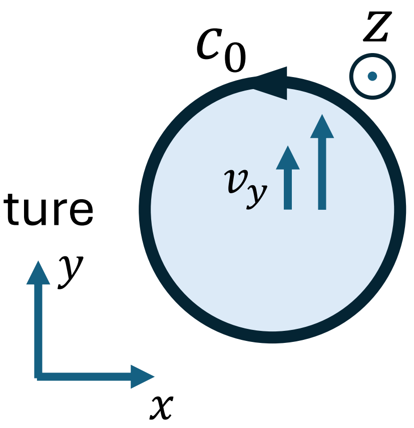
$$1/\tau_{de}=\left(\frac{D\bar{v}_y'}{3L_y^2}\right)^{1/3}$$

- Eddy tilting amplifies the effect of diffusion.

Looks familiar? Shear enhanced homogenization!

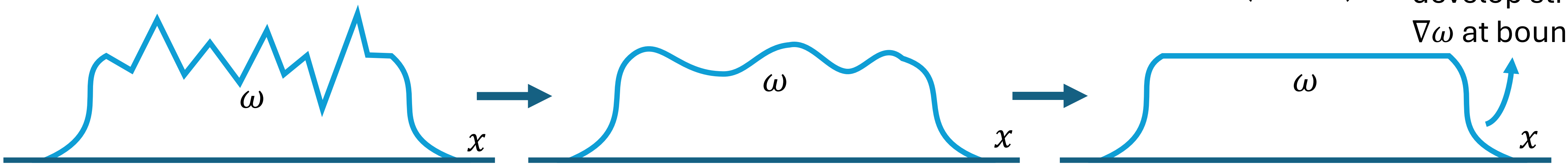
Side story: velocity shear

- For an isolated simply connected domain of 2D incompressible flow enclosed by a closed streamline $$\frac{\partial \omega}{\partial t} + \nabla \phi \times \hat{z} \cdot \nabla \omega - \nabla \cdot \nu \nabla \omega = 0$$
  when $\nu \rightarrow 0, \omega = \omega(\phi)$ is the static solution→ allows arbitrarily fine-scale structure
  Prandtl and Batchelor: when $\nu \neq 0$, the final state is $\omega(\phi) \rightarrow const.$



- What is the rate of homogenization?

$$\frac{dy}{dt} = v_y(r) \quad \frac{d\delta y}{dt} = v_y' \delta r \rightarrow \langle \delta y^2 \rangle \cong {v_y'}^2 \langle \delta r \rangle^2 t^2 \cong {v_y'}^2 D_r t^3/3 \rightarrow 1/\tau_{mix} \sim \left(\frac{{v_y'}^2 D_r}{3L_y^2} \right)^{1/3}$$



develop strong $\nabla \omega$ at boundary

initial state

‘fast mixing’ to band-like structures

‘slow mixing’ to homogenization

Thank you