Propagation of Energetic Particles Through Heliosphere

Exact Solution of Fokker-Planck equation

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Abstract / Objectives

Abstract The Fokker-Planck (FP) equation

$$\partial_t f + \mu \partial_x f = \partial_\mu \left(1-\mu^2\right) \partial_\mu f \quad (1)$$

is solved analytically. Among other applications, it describes propagation of energetic particles through a scattering medium in x- direction, with μ being the x- projection of their normalized velocity. The solution is found in terms of an infinite series of mixed moments of particle distribution, $M_{jk}(t) = \left\langle \mu^j x^k \right\rangle$. The basic moment

$$\left\langle x^2 \right\rangle = \left\langle x^2 \right\rangle_0 + t/3 + \left[\exp(-2t) - 1\right]/6 \quad (2)$$

distills a transition from ballistic (rectilinear) propagation phase, $\left\langle x^2 \right\rangle - \left\langle x^2 \right\rangle_0 \approx t^2/3$ to a time-asymptotic, diffusive phase, $\left\langle x^2 \right\rangle - \left\langle x^2 \right\rangle_0 \approx t/3$. The present paper provides all the higher moments by a recurrent formula. The full set of moments is equivalent to the full solution of FP equation, expressed in form of an infinite series in moments $\left\langle \mu^j x^k \right\rangle$. An explicit, easy-to-use approximation for a point source spreading of a pitch-angle averaged distribution $f_0(x.t)$ (starting from $f_0(x,0) = \delta(x)$, i.e. Green's function), is presented and verified by a numerical integration of FP equation.

Questions to Answer At times much shorter than the collision time, $t \ll 1$, most particles propagate with their initial velocities or their projections on the magnetic field direction, if present. This regime is called the ballistic, or rectilinear propagation. The question then is what happens next, namely at $t \sim 1$ but before the onset of diffusion at $t \gg 1$? What exactly is the value of $t > 1$, when it is safe to switch to the simple diffusive description?

Solution of FP Equation

The Fokker-Planck equation (1) is in units with the particle velocity and scattering frequency $v = D(E) = 1$ (i.e., obtained via $D(E)\, t \to t$, $\quad Dx/v \to x$). Restriction at $t = 0$, $x^n f(x) \to 0$ for $|x| \to \infty$ and $n \geq 0$ (guarantees existence of all moments)

$$M_{ij}(t) = \left\langle \mu^i x^j \right\rangle = \int_{-\infty}^{\infty} dx \int_{-1}^{1} \mu^i x^j f d\mu/2 \quad (3)$$

for integer $i, j \geq 0$. Multiplying eq.(1) by $\mu^i x^j$ and integrating by parts, we obtain

$$\frac{d}{dt} M_{ij} + i(i+1) M_{ij} = j M_{i+1,j-1} + i(i-1) M_{i-2,j} \quad (4)$$

For any $i, j \geq 0$, this equation can be resolved

$$M_{ij}(t) = M_{ij}(0)\, e^{-i(i+1)t} + \int_0^t e^{i(i+1)(t'-t)} \left[j M_{i+1,j-1}(t') + i(i-1) M_{i-2,j}(t') \right] dt' \quad (5)$$

Setting $M_{00} = 1$, we relate the moment-generating function

$$f_\lambda(t) = \int_{-\infty}^{\infty} f_0(x,t)\, e^{\lambda x} dx = \sum_{n=0}^{\infty} \frac{\lambda^{2n}}{(2n)!} M_{0,2n}(t) \quad (6)$$

to the Green function $f_0 = \int f(x,\mu) d\mu/2$, as $f_0(x,0) = \delta(x)$. This can be converted into a Fourier transform by setting $\lambda = -ik$. The Green function $f_0(x,t)$

$$f_0(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk e^{ikx} \sum_{n=0}^{\infty} (-1)^n \frac{k^{2n}}{(2n)!} M_{0,2n}(t) \quad (7)$$

For $t \gg 1$, one obtains

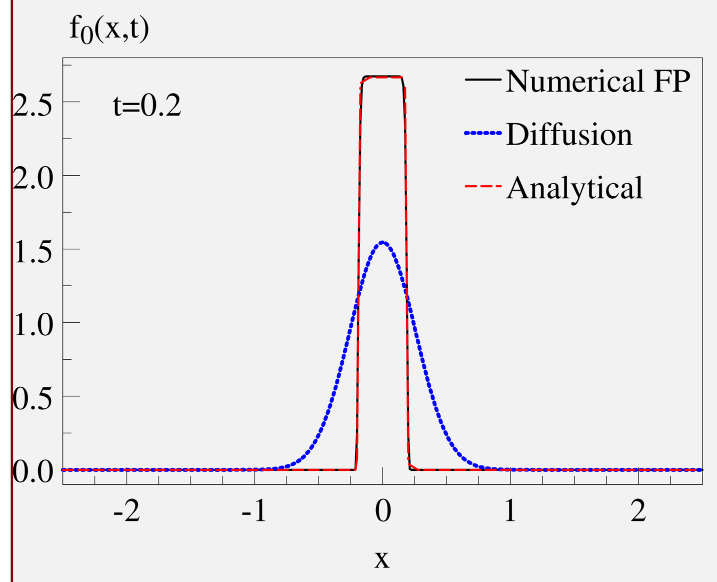
$$f_0(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk e^{ikx - k^2 t/6} = \sqrt{\frac{3}{2\pi t}} e^{-3x^2/2t} \quad (8)$$

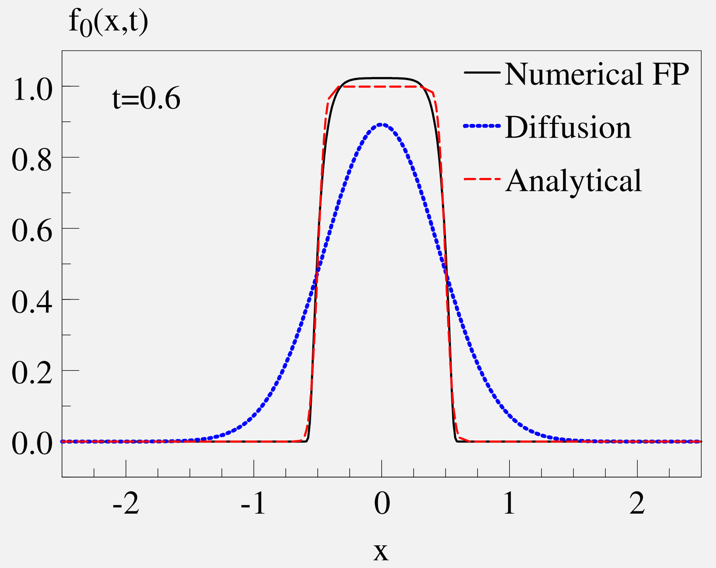
Simplified Propagator

For $t < 1$ and $t \gg 1$, by summing up the series in eq.(7) one obtains:

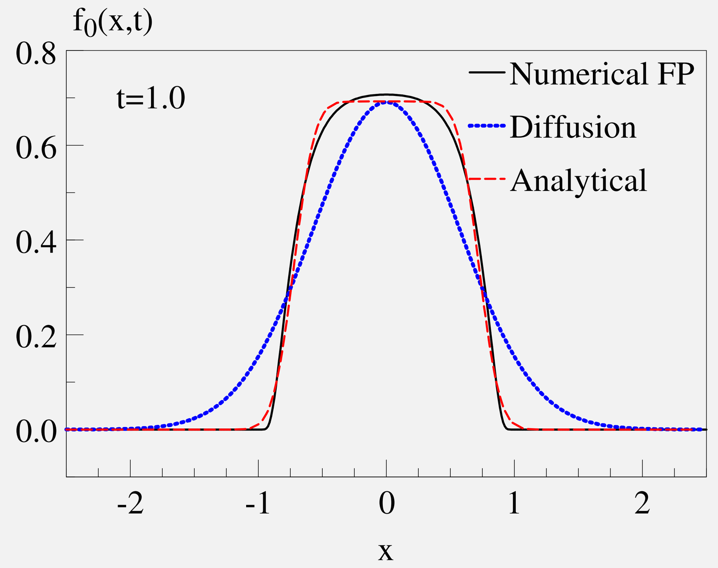
$$f_0(x,t) \approx \frac{1}{4y} \left[\operatorname{erf}\left(\frac{x+y}{\Delta}\right) - \operatorname{erf}\left(\frac{x-y}{\Delta}\right) \right] \quad (9)$$

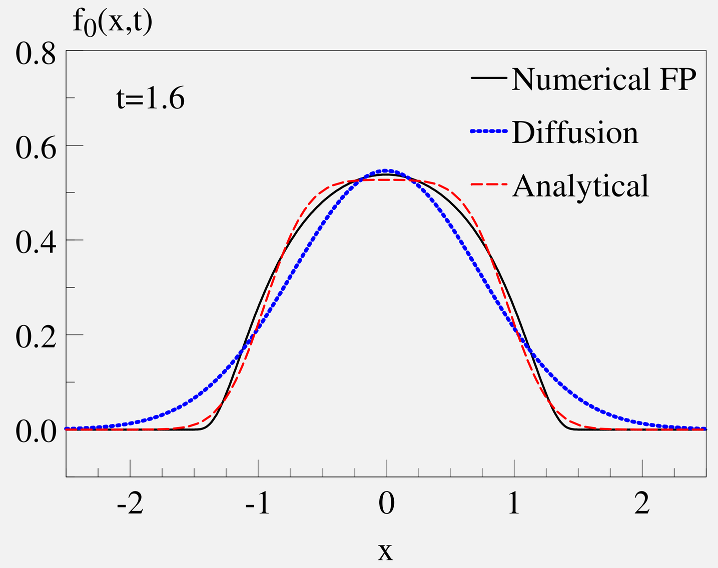
where $\Delta = 2t^2/3\sqrt{5}$, $\quad t \ll 1$, ballistic phase $\Delta = 2t^2/3\sqrt{3}$, trans-diffusive, $\Delta = \sqrt{2t/3}$, diffusive, $t \gg 1$, $y(t) \approx t$, $t \ll 1$, y=o$\left(\sqrt{t}\right)$, $t \gg 1$. Note that $\pm y(t)$ are the coordinates of two fronts propagating in opposite direction from the initial $\delta-$ pulse at $x = 0$, while $\Delta(t)$ is the front width. We observe: (1) solution in eq.(9) conserves the total number of particles, for any y and Δ, (2) recovers correct expansion for small $t < 0$, both for sharp and smooth fronts ($\lambda t > 1$, $\lambda t < 1$), (3) recovers the correct solution for $t \gg 1$

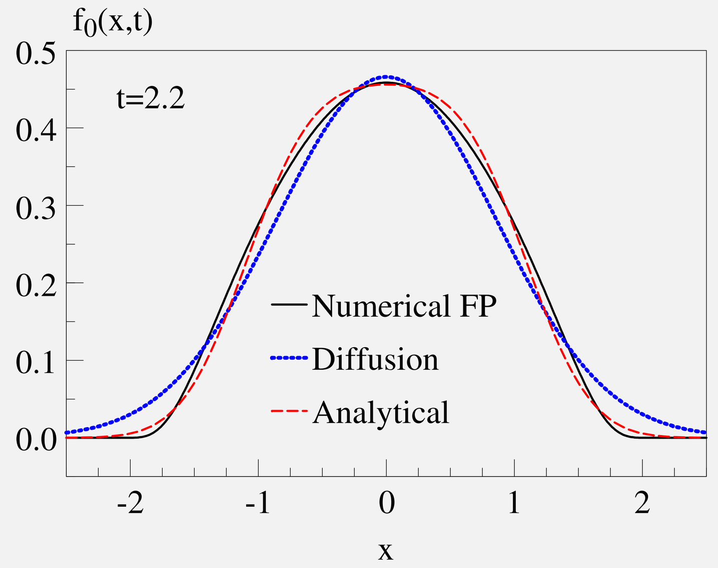


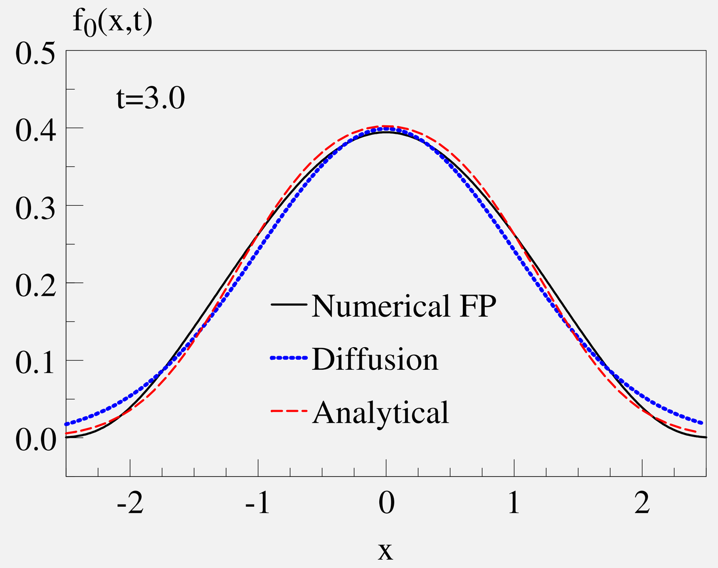


Fundamental solution of FP eq., $f_0(x,t) = \langle f(x,\mu,t) \rangle$. Analytic approximation from eq.(9).









For $t > 5$, the solution becomes diffusive, eq.(8), as predicted by Jokipii (1966).