Rossby Wave-Zonal Flow Turbulence in a Tangled Magnetic field

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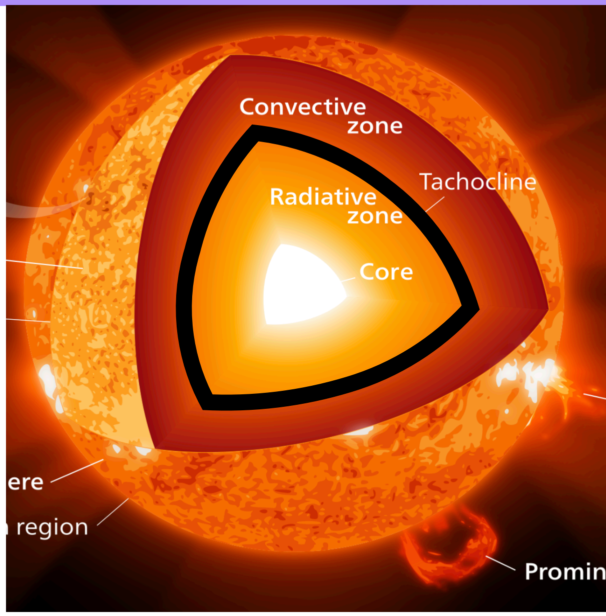
Introduction



- **Why we study the Solar Tachocline?**

  1. Driving the solar activity.

  2. Turbulences redistribute the angular momentum.

  Turbulent transport processes still poorly understood.

- **About the solar tachocline:**

  1. Between the convective and radiative zone.

  2. Strongly stratified/Pancake-like structures.

  Incompressible rotating fluid in 2D layers— β-plane model

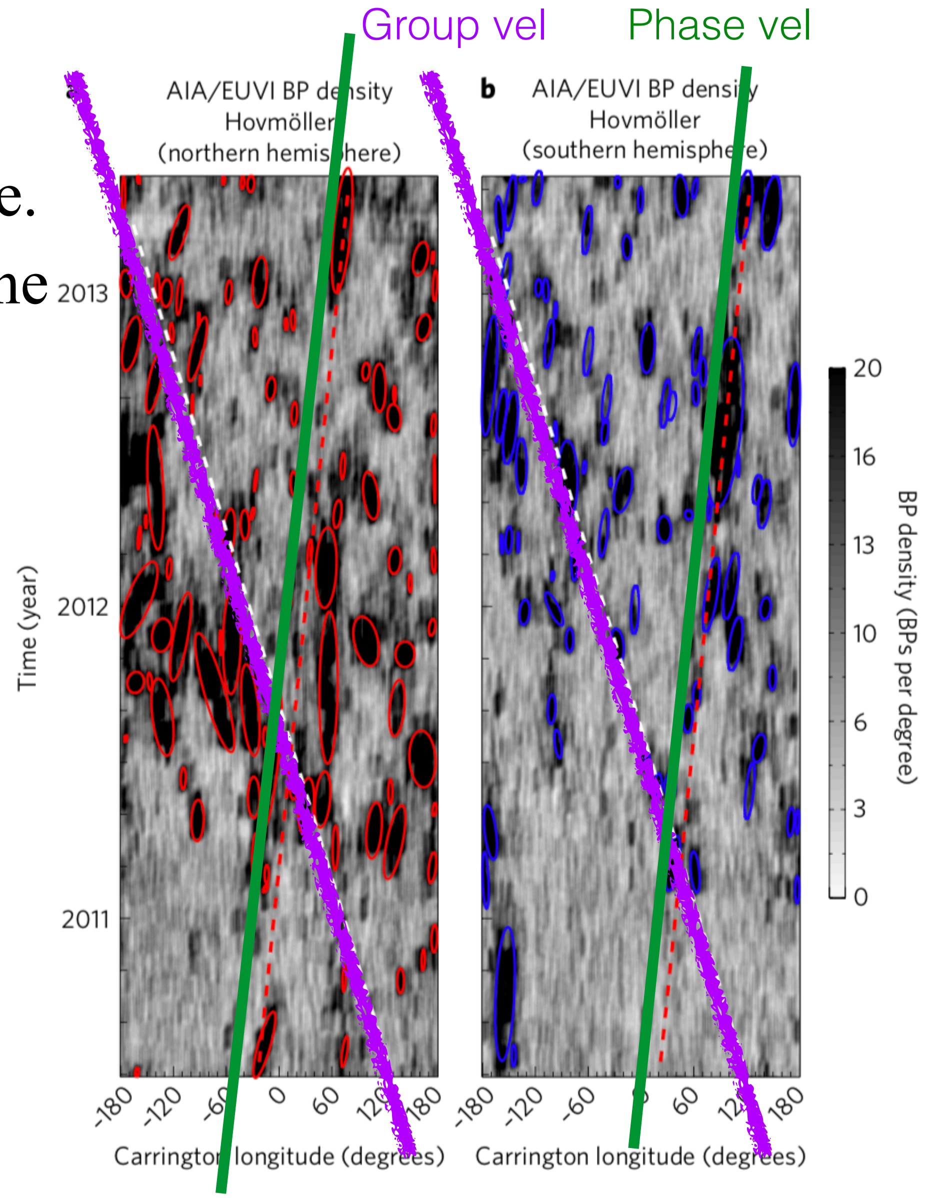
  3. Zonal Flow and Rossby Waves— as in the Jovian Atmosphere.

  Observational evidence: Magneto-Rossby-like waves on the surface of the sun Coronal brightspots (BPs) .

  4. A weak mean field— large magnetic Kubo number:

$$B = B_l + \widetilde{B}$$

$$Ku_{mag} \equiv \frac{\tilde{u}\tau_{ac}}{\Delta_{eddy}} = \frac{l_{\parallel} |\widetilde{\mathbf{B}}|}{\Delta_{\perp} B_l} \gg 1$$

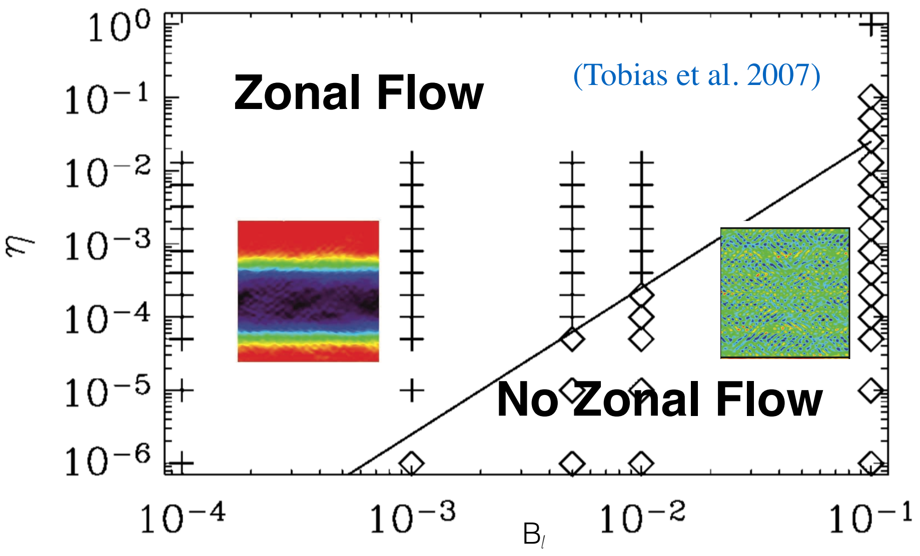


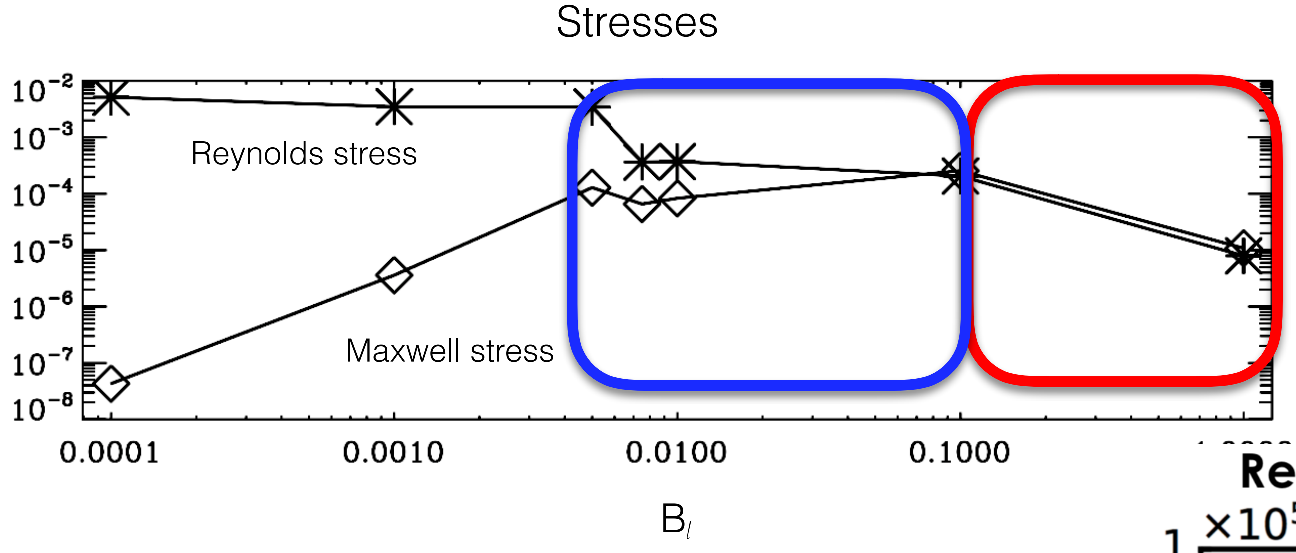
(McIntosh et al. 2017)

Introduction— Takeaways

- **Studies have shown:**

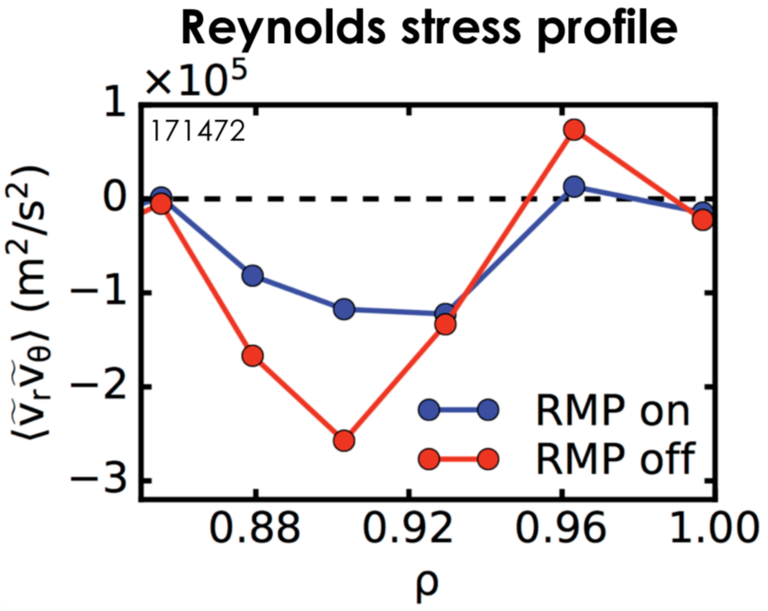
  1. Simulation of the Solar tachocline: Tobias et al. (2007) found a transition line.
  2. The **Reynolds stress** will be suppressed at levels of field intensities **well below that of Alfvènization**, where Maxwell stress balances the Reynolds stress.
  3. Related: An experiment on edge of DIIID— RMP alters the Reynolds stress and increases the threshold of LH transition.





- **We obtained:**

  1. **Dimensionless parameters** which successfully predict the transition line.
  2. A theory of how the small-scale static **random magnetic fields suppress** the Reynolds stress.
  3. This suppression happens when mean magnetic intensities **BELOW** that of Alfvènization.



(D. M. Kriete et al., TTF 2019)

Introduction

- **Two main equations:**

$$\left.\begin{aligned}(\frac{\partial}{\partial t} + \mathbf{u}_\perp \cdot \nabla_\perp)\zeta - \beta\frac{\partial \psi}{\partial x} &= -\frac{(\mathbf{B}\nabla)(\nabla^2 A_z)}{\mu_0 \rho} + \nu \nabla^2 \zeta \\ (\frac{\partial}{\partial t} + \mathbf{u}_\perp \cdot \nabla_\perp)A &= B_l \frac{\partial \psi}{\partial x} + \eta \nabla^2 A,\end{aligned}\right\}$$

2D MHD:
Alfvèn-Rossby Waves on
β-plane

- **Rossby Parameter (β) and the β-plane model:**

$$\beta = \frac{df}{dy}|_{\phi_0} = 2\Omega cos(\phi_0)/a$$

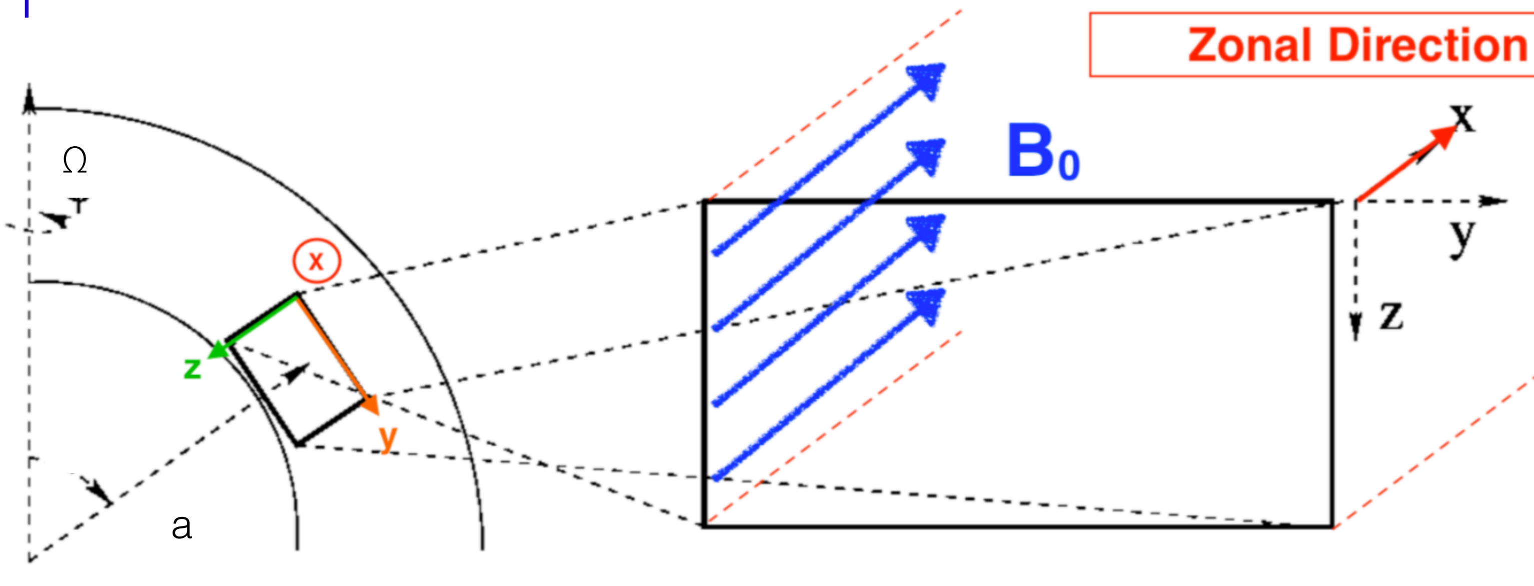
rotation

latitude

radius

Derivative of angular frequency f
(Coriolis parameter)

- **Two limits:**
1. Small magnetic Kubo number
(strong mean field)
2. Large magnetic Kubo number
(weak mean field)



A Model for PV Transport in Strong mean Magnetic field

Model— Large mean field: Notations

- **Notations we have:**

Stream Function $\psi = \psi(x, y, z)$

Velocity field $\boldsymbol{u} = (\frac{\partial \psi}{\partial y}, -\frac{\partial \psi}{\partial x}, 0)$

Fluid Vorticity $\boldsymbol{\zeta} = (0, 0, \zeta)$

Potential Field $\boldsymbol{A} = (0, 0, A)$

Magnetic Field $\boldsymbol{B} = (\frac{\partial A}{\partial y}, -\frac{\partial A}{\partial x}, 0),$

- **Two main equations:**
  - **QL closure**
  - **linear response of perturbations**

$$\begin{cases} (\frac{\partial}{\partial t} + \mathbf{u}_{\perp} \cdot \nabla_{\perp})\zeta - \beta \frac{\partial \psi}{\partial x} = -\frac{(\mathbf{B}\nabla)(\nabla^2 A_z)}{\mu_0 \rho} + \nu(\nabla \times \nabla^2 \mathbf{u}) \\ (\frac{\partial}{\partial t} + \mathbf{u}_{\perp} \cdot \nabla_{\perp})A = B_l \frac{\partial \psi}{\partial x} + \eta \nabla^2 A, \end{cases}$$

$$\begin{cases} \widetilde{\zeta_k} = \left(\frac{i}{\omega + i\nu k^2 + (\frac{B_l^2}{\mu_0 \rho}) \frac{k_x^2}{-\omega - i\eta k^2}} \right) (\widetilde{u_y} \frac{-\partial}{\partial y} \langle \zeta \rangle - \beta \widetilde{u_y}) \\ \widetilde{A_k} = \frac{\widetilde{\zeta_k}}{k^2} (\frac{B_l k_x}{-\omega - i\eta k^2}) \end{cases}$$

Quasi-Linear Approximation:

$$\begin{array}{r|c|} \cline{2-2} \zeta = \langle \zeta \rangle + & \widetilde{\zeta} \\ \psi = \langle \psi \rangle + & \widetilde{\psi} \\ A = \langle A \rangle + & \widetilde{A} \\ \cline{2-2} \end{array}$$

Perturbations produced by turbulences

$$, where \ \langle \ \rangle = \frac{1}{L} \int dx \frac{1}{T} \int dt$$

ensemble average over the zonal scales

Model— Large mean field: Notations

- **PV flux:**
  - ➤ **By using Taylor Identities we have the Reynolds and Maxwell stress**
  - ➤ **Express the PV flux with two diffusivities**

$$\frac{\partial}{\partial t}\langle\zeta_k\rangle = -\frac{\partial}{\partial y}\left(\langle\widetilde{u}_{y,k}\widetilde{\zeta_k}\rangle + \frac{\langle\widetilde{B}_{y,k}\nabla^2\widetilde{A}_k\rangle}{\mu_0\rho}\right) \equiv -\frac{\partial}{\partial y}\Gamma,$$

$$\frac{\partial}{\partial t}\langle v_x\rangle = -\frac{\partial}{\partial y}\left\{\langle\widetilde{v}_{x,k}\widetilde{v}_{y,k}\rangle - \frac{\langle\widetilde{B}_{x,k}\widetilde{B}_{y,k}\rangle}{\mu_0\rho}\right\} = \Gamma$$

by Taylor Identity

$$\Gamma = \langle\widetilde{u}_y\widetilde{\zeta}\rangle = (-D_{fluid} + D_{mag})\frac{\partial}{\partial y}\langle PV\rangle$$

PV flux

Diffusivity
of potential vorticity

$$\begin{cases} D_{fluid} = \sum_k |\widetilde{u}_{y,k}|^2 \dfrac{\nu k^2 + \omega_A^2\frac{\eta k^2}{\omega^2+\eta^2k^4}}{\left(\omega - \omega_A^2\frac{\omega}{\omega^2+\eta^2k^4}\right)^2 + \left(\nu k^2 + \omega_A^2\frac{\eta k^2}{\omega^2+\eta^2k^4}\right)^2} \\ D_{mag} = \sum_k |\widetilde{u}_{y,k}|^2 \dfrac{\omega_A^2\left(\nu k^2(\omega^2+\eta^2k^4) + \omega_A^2\eta k^2\right)}{\omega^2\left(\omega^2+\eta^2k^4-\omega_A^2\right)^2 + \left(\nu k^2(\omega^2+\eta^2k^4)+\omega_A^2\eta k^2\right)^2}. \end{cases}$$

$$\times\left(\frac{B_{0,i}^2}{\mu_0\rho}\right)\frac{k_i^2}{\omega^2+\eta^2k^4}$$

Model— Large mean field: Results

- **Strong Mean Field:**

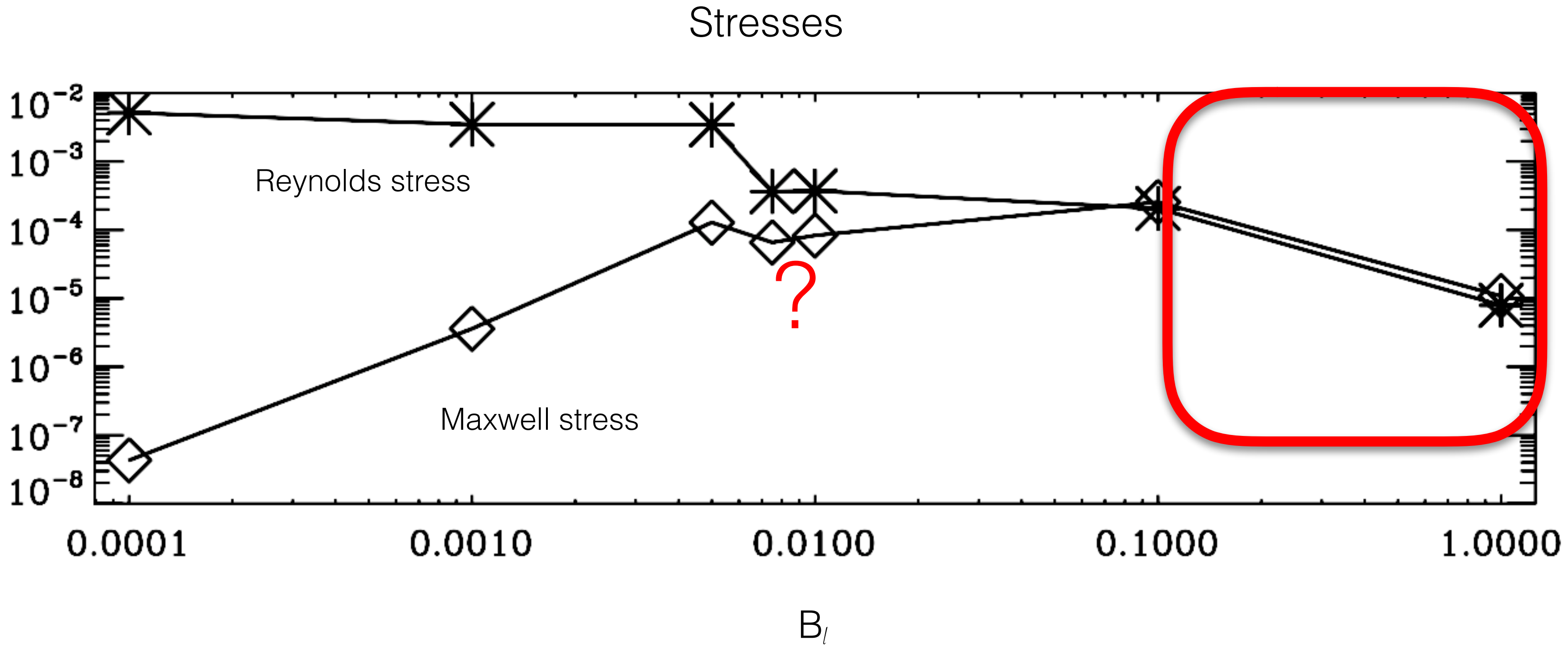
For the strong large-scale field: $\left(\omega \sim \sqrt{\frac{B_0^2}{\mu_0\rho}}k_j \gg \eta k^2, \nu k^2, \omega_R\right)$

$$D_{fluid} = D_{mag}$$

$$\Rightarrow \frac{\partial}{\partial y}\langle \widetilde{u_x}\widetilde{u_y}\rangle = \frac{\partial}{\partial y}\frac{\langle \widetilde{B_x}\widetilde{B_y}\rangle}{\mu_0\rho}$$

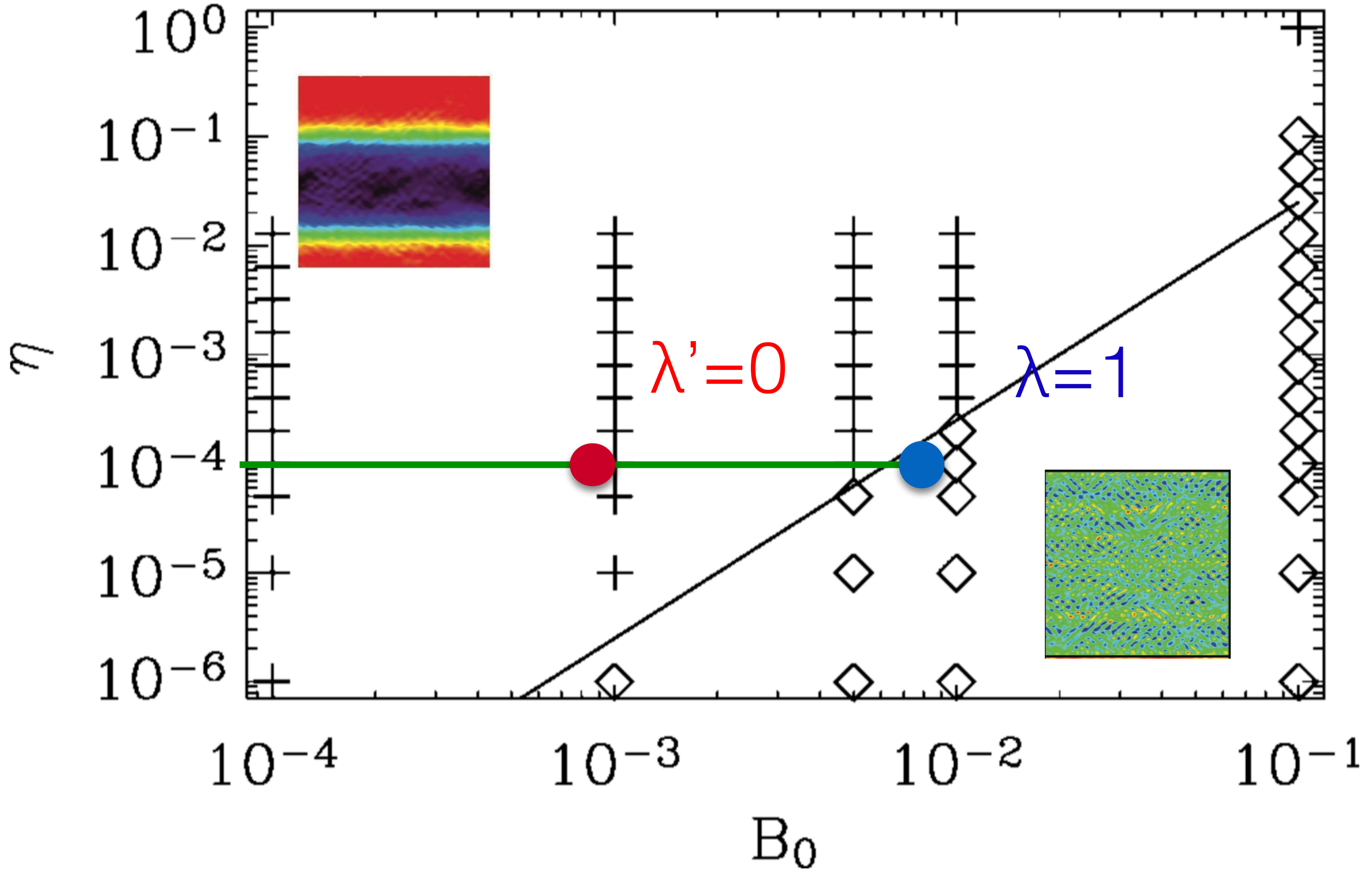
$$\Rightarrow \langle \Gamma \rangle = 0 + \mathcal{O}\left(\frac{\eta k^2}{\sqrt{\frac{B_0^2}{\mu_0\rho}}k_j}\right)^2$$

The MHD turbulence plays no role in transporting momentum when the system is fully Alfvènized.



Model— Large mean field: Results and Predictions

- **Two KEY dimensionless parameters:**



(Tobias et al. 2007)

Critical growth parameter (λ'):

$$\left[\lambda' \equiv \frac{D_{fluid} - D_{mag}}{D_{fluid}}\right]$$

if $\lambda' = 0$, zonal flow stops growing

(zonal flow production saturated).

$$\lambda^{'} = 1 - \frac{B_{0,i}^2 k_i^2}{\mu_0 \rho} / (\omega^2 + \eta^2 k^4)$$

Transition parameter (λ):

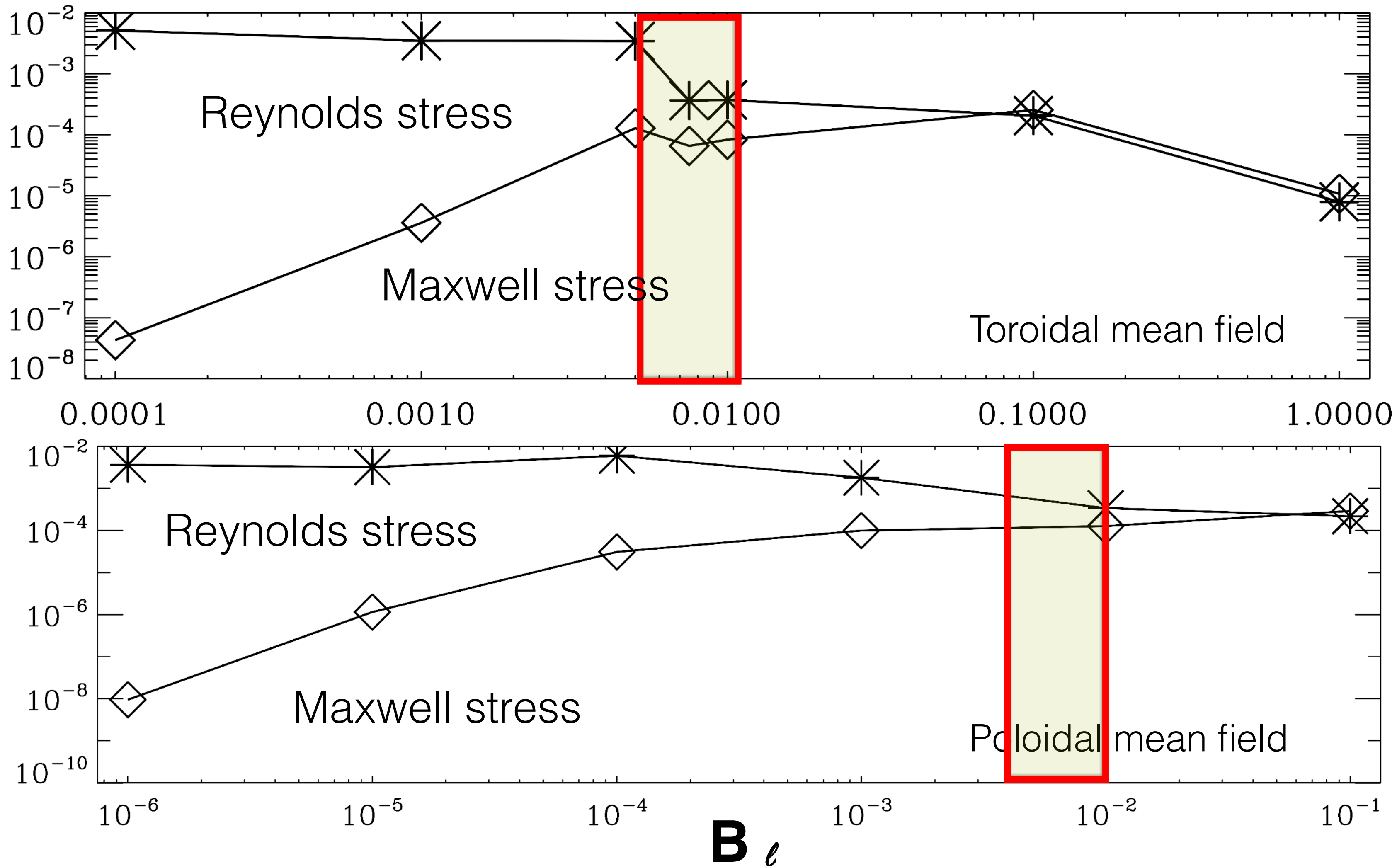
$$\left[\lambda = \frac{\omega_{im}}{\omega_{re}}\right]$$

if $\lambda = 1$, the wave is critically damped.

$$\lambda \equiv |\frac{\omega_{im}}{\omega_{re}}| = \frac{\eta k^2(\omega_R - \sqrt{\omega_R^2 + 4\omega_A^2})}{4\omega_A^2 \sqrt{\omega_R^2 + 4\omega_A^2}}$$

Model— Large mean field: Results and Predictions

- **Predictions of the transition (λ) regime:**



(Tobias et al. in preparation)

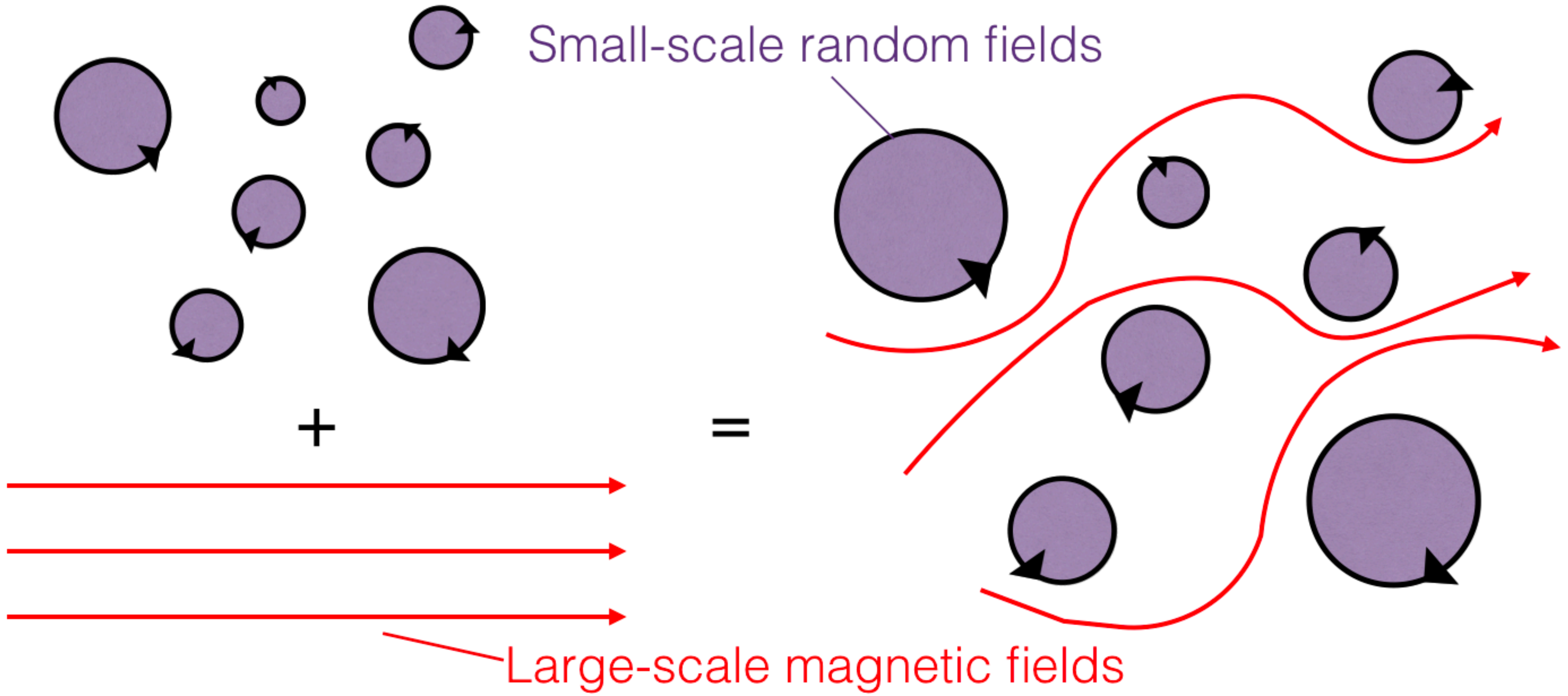
The Reynolds stress drops by an order of magnitude in the regime (toroidal mean field case) we predicted.

This occurs at weaker B_ℓ than that for which the system is fully Alfvènized! →

The cross phase effect suppresses the Reynolds stress when mean field is weak!

A Model for PV Transport in Random, Small-scale Magnetic fields

Model— Random fields: Order of Scales



- **Properties of random fields:**
  1. Smaller scale
  2. Static
  3. Randomness in space
  4. Amplitudes of random fields distributed statistically.
  (assumption: PDF Gaussian)

→

Large magnetic Kubo number:

$$Ku_{mag} \equiv \frac{\tilde{u}\tau_{ac}}{\Delta_{eddy}} = \frac{l_{\|} \, |\widetilde{\mathbf{B}}|}{\Delta_{\perp} B_l} > 1$$

Model— Random fields: Order of Scales

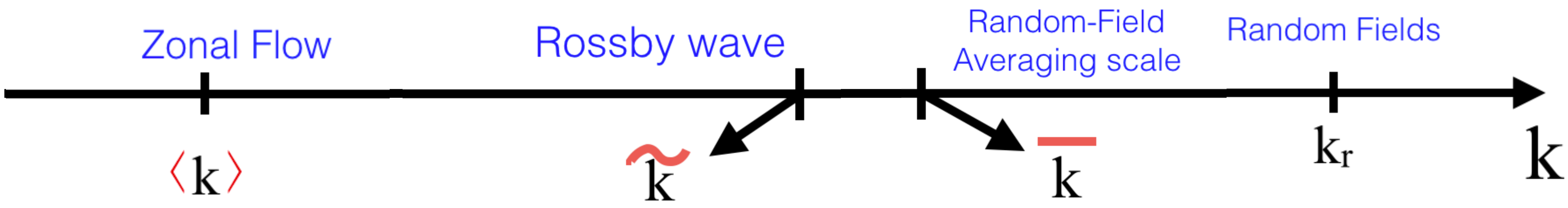
potential field	$\mathbf{A} = \mathbf{A}_\mathbf{l} + \widetilde{\mathbf{A}} + \mathbf{A}_\mathbf{r}$
magnetic field	$\mathbf{B} = \mathbf{B}_\mathbf{l} + \widetilde{\mathbf{B}} + \mathbf{B}_\mathbf{r}$
magnetic current	$\mathbf{J} = \mathbf{0} + \widetilde{\mathbf{J}} + \mathbf{J}_\mathbf{r},$
stream function	$\psi = \langle \psi \rangle + \widetilde{\psi}$
flow velocity	$\mathbf{u} = \langle \mathbf{u} \rangle + \tilde{\mathbf{u}}$
vorticity	$\zeta = \langle \zeta \rangle + \tilde{\zeta}$

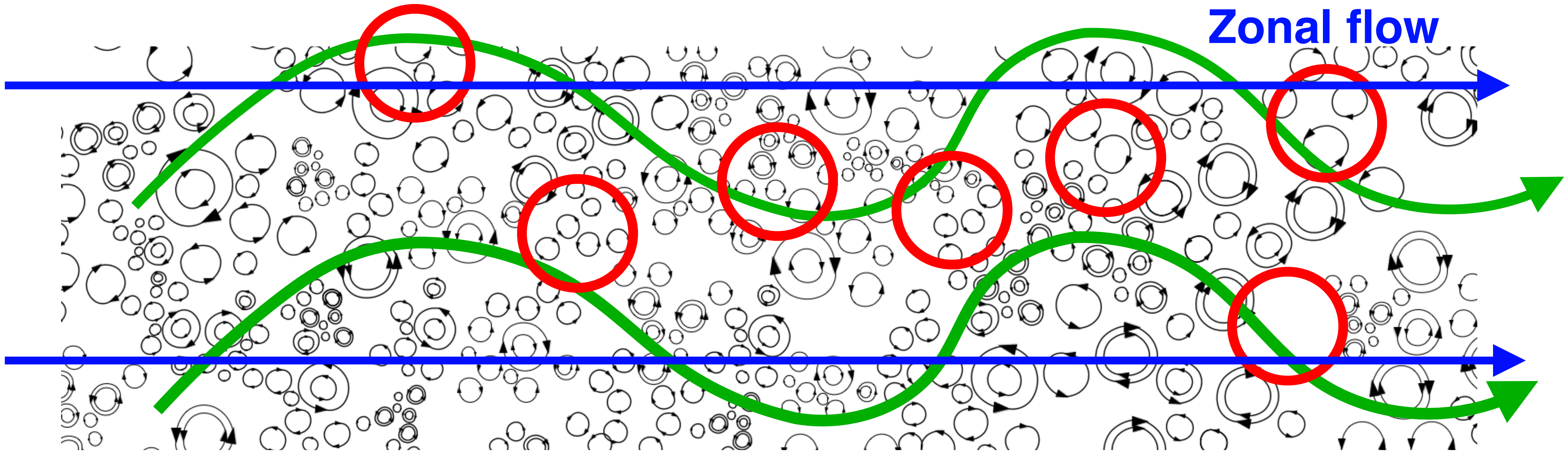
Two-average Method:

1.
$$\overline{F} = \int dR^2 \int dB_r \cdot P_{(B_{r,x},B_{r,y})} F$$

2.
$$\langle \rangle = \frac{1}{L}\int dx \frac{1}{T}\int dt$$

ensemble average over the zonal scales





Model— Random fields: Assumptions and Results

- **Assumptions:**

1. $\overline{B_{r,i}} = 0$ The averaging scale we chose

   $\overline{B_{r,x}B_{r,y}} = 0$ We approximate the correlation matrix as diagonal (TBD) ,

2. The collective field at Rossby-scale is NOT large enough to alter the structure of the random fields: $\widetilde{B_r} \to 0$

Two main equations:

$$\begin{cases} \frac{\partial}{\partial t}\bar{\zeta} - \beta\frac{\partial\bar{\psi}}{\partial x} = \boxed{\frac{\overline{(\mathbf{B}\cdot\nabla)J}}{\mu_0\rho}} + \nu\nabla^2\bar{\zeta} \\ \frac{\partial}{\partial t}A = \mathbf{B}\cdot\nabla\psi + \eta\nabla^2 A\,. \end{cases}$$

Key term:
Average effect of J x B

Linear response of the vorticity:

$$\widetilde{\zeta_k} = \left(\frac{i}{\omega + i\nu k^2 + \frac{i\overline{B^2_{r,j}}k_j^2}{\mu_0\rho\eta k^2} + \frac{i}{\mu_0\rho}\frac{B^2_{l,x}k_x^2}{\eta k^2 - i\omega}}\right)\widetilde{u}_{y,k}\left(-\frac{\partial}{\partial y}\bar{\zeta} - \beta\right)$$

- **Dispersion relation of the Rossby-Alfvén wave in random magnetic fields:**

$$\left(\omega - \omega_R + \boxed{\frac{i\overline{B^2_{r,y}}k_y^2}{\mu_0\rho\eta k^2}} + i\nu k^2\right)\left(\omega + i\eta k^2\right) = \frac{B^2_{l,x}k_x^2}{\mu_0\rho}$$

(square mean)

Dissipative response to
Random magnetic fields

(mean square)

AW of the large-scale
magnetic field

Random fields: Results— Suppression

- **Flux and diffusivity of potential vorticity:**

$$\overline{\Gamma} = -\boxed{\sum_k |\widetilde{u}_{y,k}|^2 \frac{\nu k^2 + (\frac{B_l^2 k_x^2}{\mu_0 \rho})\frac{\eta k^2}{\omega^2 + \eta^2 k^4} + \frac{\overline{B_{r,y}^2} k_y^2}{\mu_0 \rho \eta k^2}}{\left(\omega - (\frac{B_l^2 k_x^2}{\mu_0 \rho})\frac{\omega}{\omega^2 + \eta^2 k^4}\right)^2 + \left(\nu k^2 + (\frac{B_l^2 k_x^2}{\mu_0 \rho})\frac{\eta k^2}{\omega^2 + \eta^2 k^4} + \frac{\overline{B_{r,y}^2} k_y^2}{\mu_0 \rho \eta k^2}\right)^2}}(\frac{\partial}{\partial y}\overline{\zeta} + \beta)$$

PV Diffusivity **D**

Large-scale field

small-scale random fields

$$\bar{\Gamma} = -D(\frac{\partial \bar{\zeta}}{\partial y} + \beta)$$

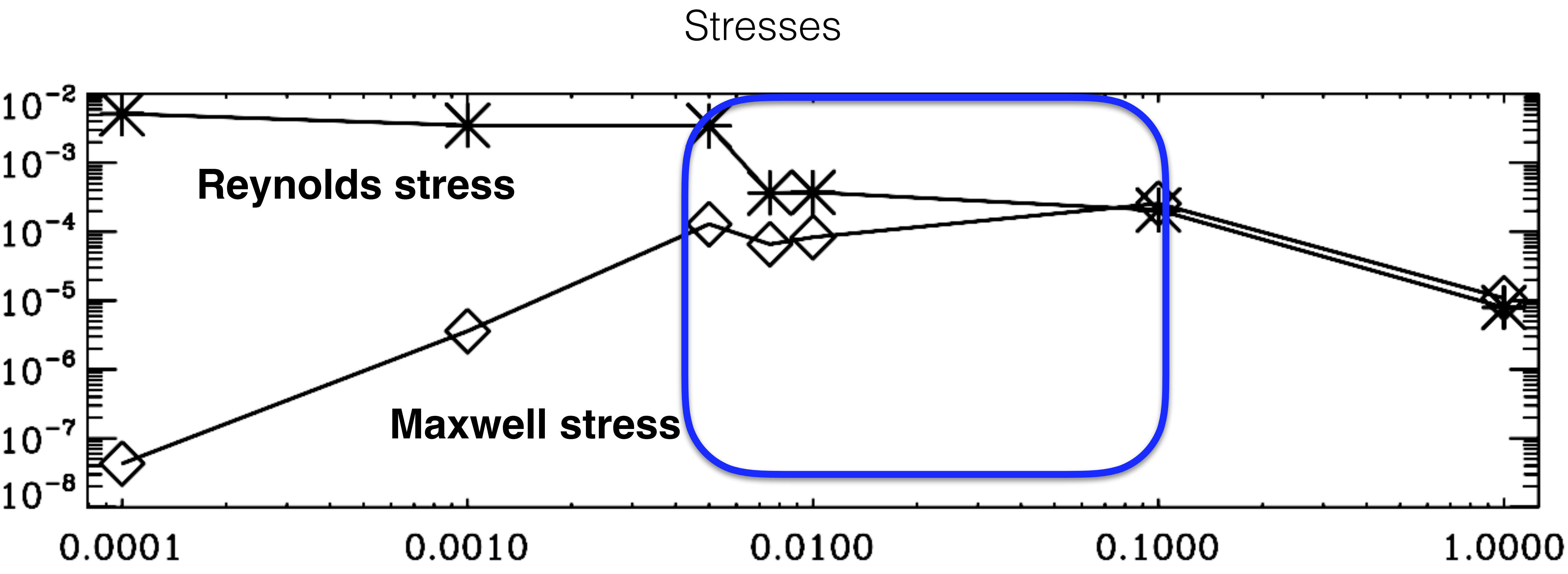
The large- and small-scale magnetic fields have a synergistic effect on the cross-phase in the Reynolds stress.

In a certain limit where $B_l^2 \ll \overline{B_{r,y}^2}$:

$$D = \sum_k |\widetilde{u}_{y,k}|^2 \frac{\nu k^2 + \frac{\overline{B_{r,y}^2} k_y^2}{\mu_0 \rho \eta k^2}}{\omega^2 + \left(\nu k^2 + \frac{\overline{B_{r,y}^2} k_y^2}{\mu_0 \rho \eta k^2}\right)^2}$$

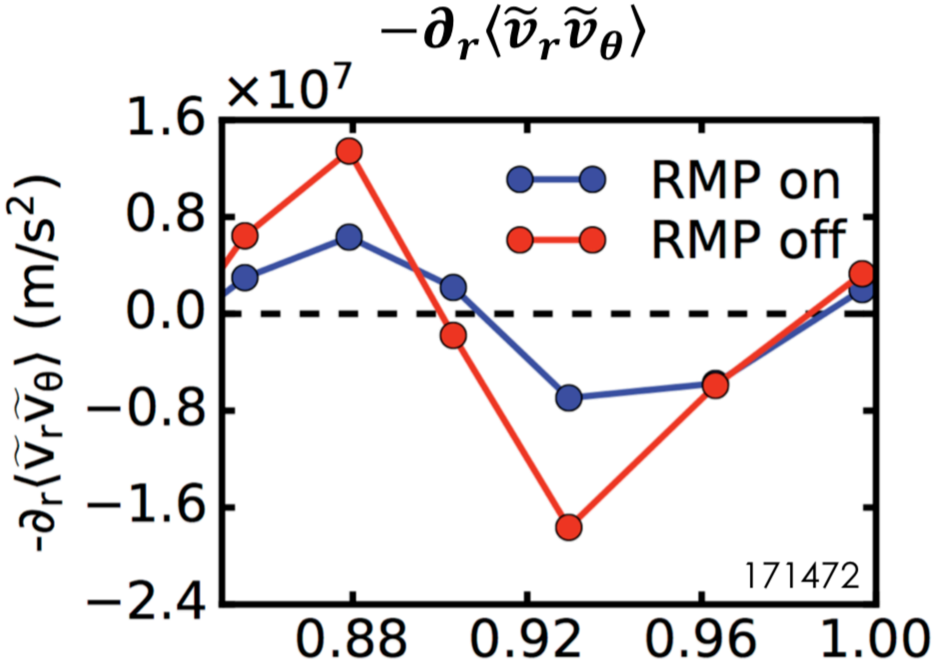
Cross phase drops for stronger small-scale random fields!

Random fields: cont'd— Suppression



β_l

$\bar{\Gamma}$



(D. M. Kriete et al., TTF 2019)

A related *Experiment (DIII-D):* Resonant Magnetic Perturbations (RMP) suppress the Reynolds stress, increasing the threshold of the L-H transition.

Random fields: Results— Zonal flow evolution

- **Evolution of Zonal Flow**

➤ Random magnetic fields suppress the Reynolds stress and increase the drag.

$$\frac{\partial}{\partial t}\langle u_x\rangle = \langle\overline{\Gamma}\rangle - \frac{1}{\eta\mu_0\rho}\langle\overline{B^2_{r,y}}\rangle\langle u_x\rangle + \nu\nabla^2\langle u_x\rangle$$

Cross-phase effect

Magnetic drag force $(J_r \times B_r)$

Random magnetic fields have an effect on both the PV flux and the magnetic drag.

$$\overline{\Gamma} = -\sum_k |\tilde{u}_{y,k}|^2 \frac{\nu k^2 + (\frac{B_l^2k_x^2}{\mu_0\rho})\frac{\eta k^2}{\omega^2+\eta^2k^4} + \frac{\overline{B^2_{r,y}}k_y^2}{\mu_0\rho\eta k^2}}{\left(\omega - (\frac{B_l^2k_x^2}{\mu_0\rho})\frac{\omega}{\omega^2+\eta^2k^4}\right)^2 + \left(\nu k^2 + (\frac{B_l^2k_x^2}{\mu_0\rho})\frac{\eta k^2}{\omega^2+\eta^2k^4} + \frac{\overline{B^2_{r,y}}k_y^2}{\mu_0\rho\eta k^2}\right)^2}\left(\frac{\partial}{\partial y}\overline{\zeta} + \beta\right),$$

small-scale random fields

Model— Random fields: Results— key parameters

$$(\omega_R \sim \omega_{re} > \omega_{B_r} \gg \eta k^2 \gg \nu k^2 \sim \omega_A^2)$$

Critical growth parameter (λ’):

$$\lambda' \equiv \frac{\langle\overline{\Gamma}\rangle - \frac{1}{\eta\mu_0\rho}\langle\overline{B^2_{r,y}}\rangle\langle u_x\rangle}{\langle\overline{\Gamma}\rangle}$$

if $\lambda' = 0$, the zonal flow stop growing.

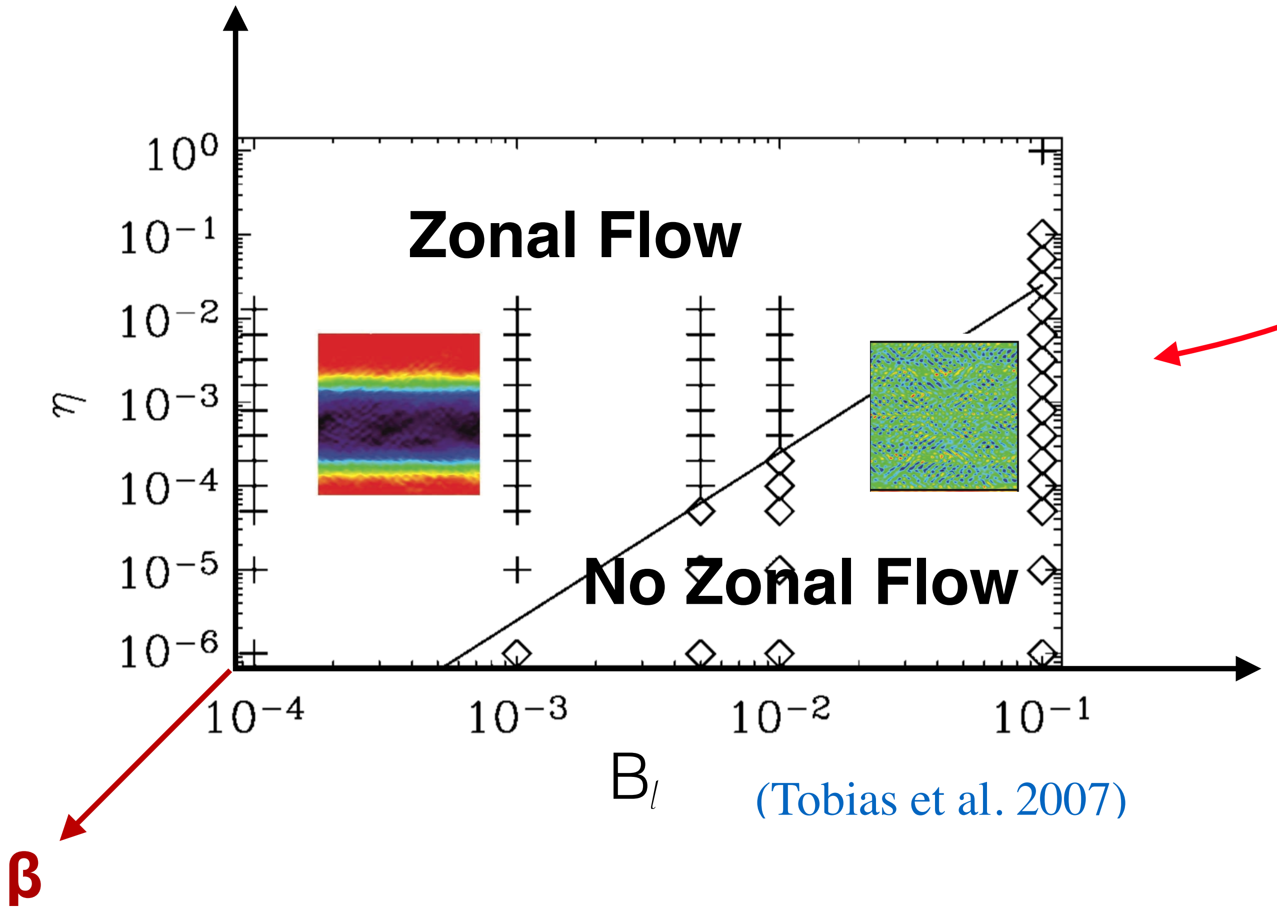
Numerically, we need ⟨ $\overline{B_{r,y}^2}$ ⟩

Transition parameter (λ):

$$\left[\lambda = \frac{\omega_{im}}{\omega_{re}}\right]$$

if $\lambda = 1$, the wave is critically damped.

$$\lambda \equiv \left|\frac{\omega_{im}}{\omega_{re}}\right| = \frac{\omega_{Br}^2}{\eta k^2 \omega_R} = \frac{\omega_{Br}^2}{\eta k_x \beta} \propto \boxed{\frac{\overline{B_r^2}}{\eta\beta}}$$



(Tobias et al. 2007)

Random fields: Physical picture— Effective medium

- **Resisto-Elastic medium:**

If we turnoff the Rossby frequency, we have a 2D non-rotating plane:

$$\omega^2 + i(\alpha + \eta k^2)\omega - (\alpha\eta k^2 + \chi) = 0$$

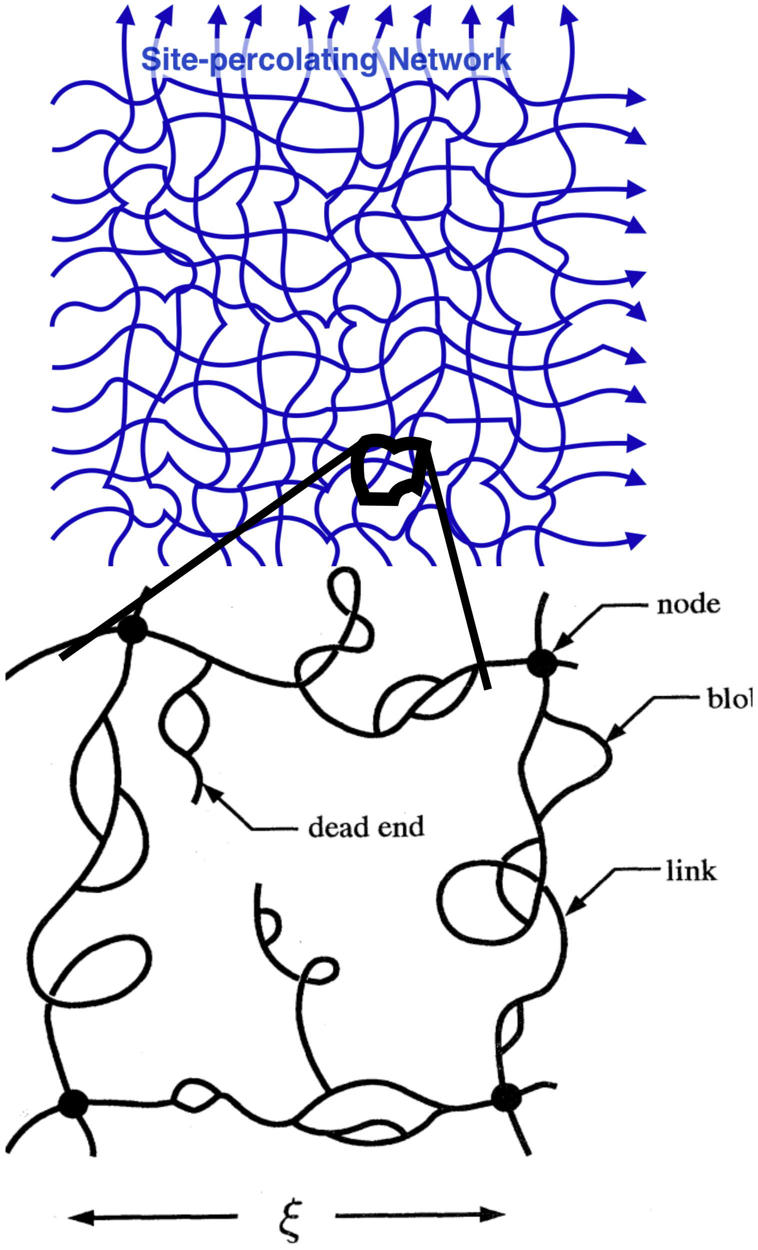
drag + dissipation

effective spring constants

$$\alpha \equiv \frac{\overline{B_{r,j}^2} k_j^2}{\mu_0 \rho \eta k^2} \propto \frac{\text{spring constant}}{\text{dissipation}} \qquad \chi \equiv \frac{B_{l,x}^2 k_x^2}{\mu_0 \rho}$$

→ **resisto-elastic** medium!

- **More can be done:**
Fractal Network (Site-percolating) —
Calculate the effective spring constant, effective Young's Modulus of elasticity, and effective "conductivity" of vorticity (such as encountered in amorphous solids).



Schematic of the nodes-links-blobs model (Nakayama & Yakubo 1994).

Random fields: Relation to 'Quenching' ?

- **Recall the suppression of turbulent magnetic resistivity (η_T) by weak magnetic field:**

$$D_M = \eta_T = \sum_k \frac{|\widetilde{u}_k^2|\tau_{c,k}}{1 + Rm\frac{v_{A,l}^2}{\langle\widetilde{u}^2\rangle}} \overset{\uparrow}{=} \sum_k \frac{|\widetilde{u}_k^2|\tau_{c,k}}{1 + \frac{v_{A,r}^2}{\langle\widetilde{u}^2\rangle}}$$

where $v_{A,l}^2 = \frac{B_l^2}{\mu_0\rho}$, and $v_{A,r}^2 = \frac{\overline{B_{r,y}^2}}{\mu_0\rho}$.

$$\boxed{\overline{B_{r,y}^2} \sim Rm \cdot B_l^2}$$

(Zeldovich et al. 1957)

- **We derive an expression for PV diffusivity (D):**

$$D_{PV} = \sum_k \frac{|\widetilde{u}_{y,k}|^2 \frac{\alpha}{\omega^2}}{1 + (\frac{\alpha}{\omega})^2},\quad \text{where } \alpha \equiv \frac{v_{A,r}^2 k_y^2}{\eta k^2}$$

$$\frac{\alpha}{\omega^2} \sim \tau_c$$

The effect on both D_M and D_{PV} are due to $\overline{B_{r,y}^2}$ effects, via the relation of Zeldovich.

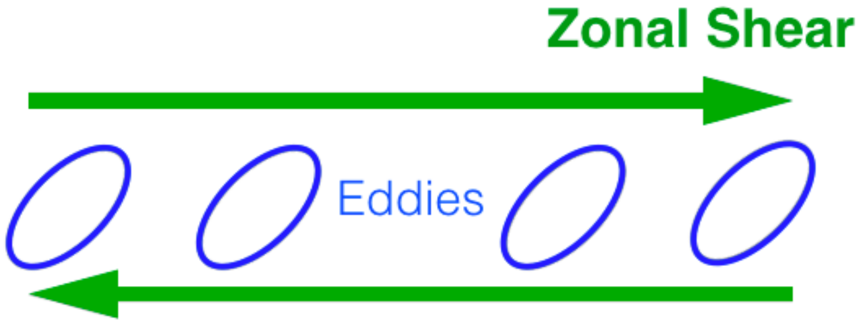
Fundamental physics are the same, though the difference in detail is due to different parameter regimes.

Random fields: Future Works

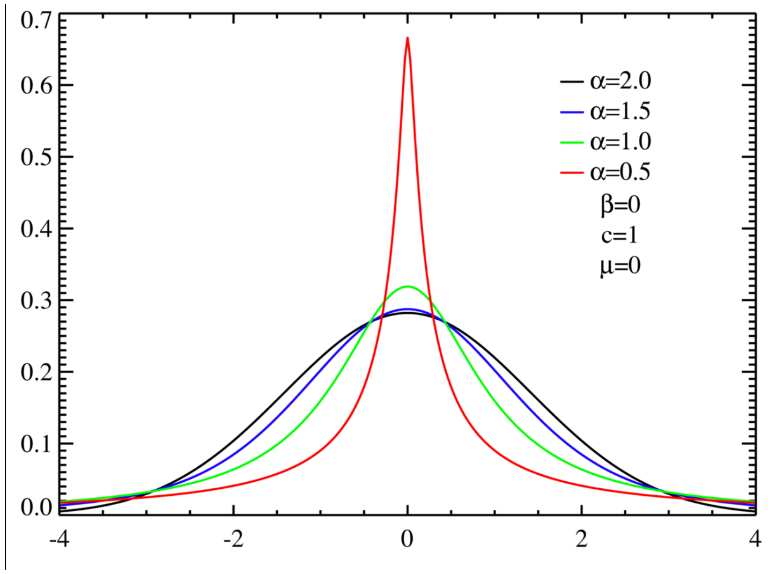
- Shear will induce the correlation even $B_{r,x}$ and $B_{r,y}$ are initially uncorrelated.

Symmetry breaking by zonal shear!

reconsider the effect of $\overline{B_{r,x}B_{r,y}} \neq 0$



- PDF of random magnetic field can have a fat tail.



- The random magnetic fields is not static.
  OR
  Reconsider the perturbation from Rossby wave turbulence.

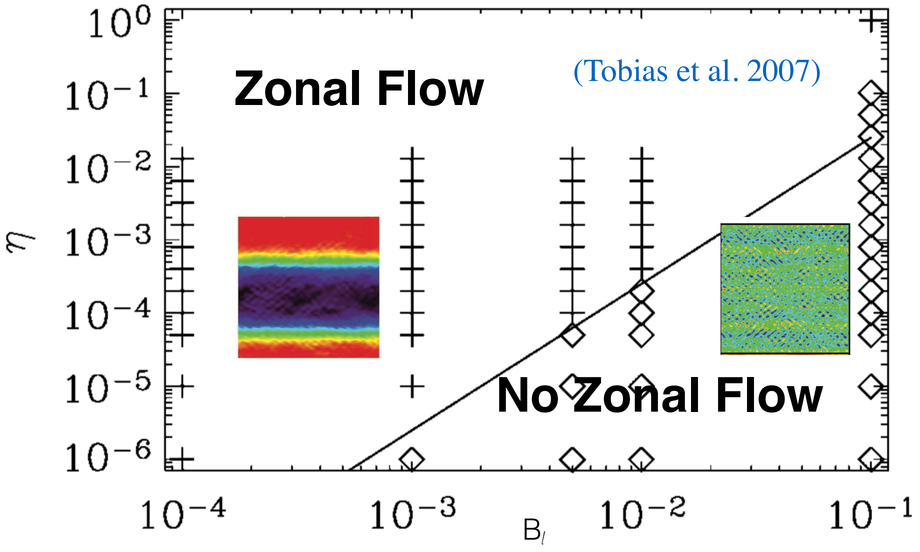
$$\widetilde{B}_r \not\rightarrow 0$$

- Revisit the magnetic diffusivity η_T.

Takeaways

- **Studies have shown:**

1. Tobias et al. (2007) found a transition line.
2. **Reynolds stress** will be suppressed at levels of field intensities **well below that of Alfvènization**, where Maxwell stress balances the Reynolds stress.
3. Related: Experiment on edge of DIIID— RMP alters the Reynolds stress and increases the threshold of LH transition.



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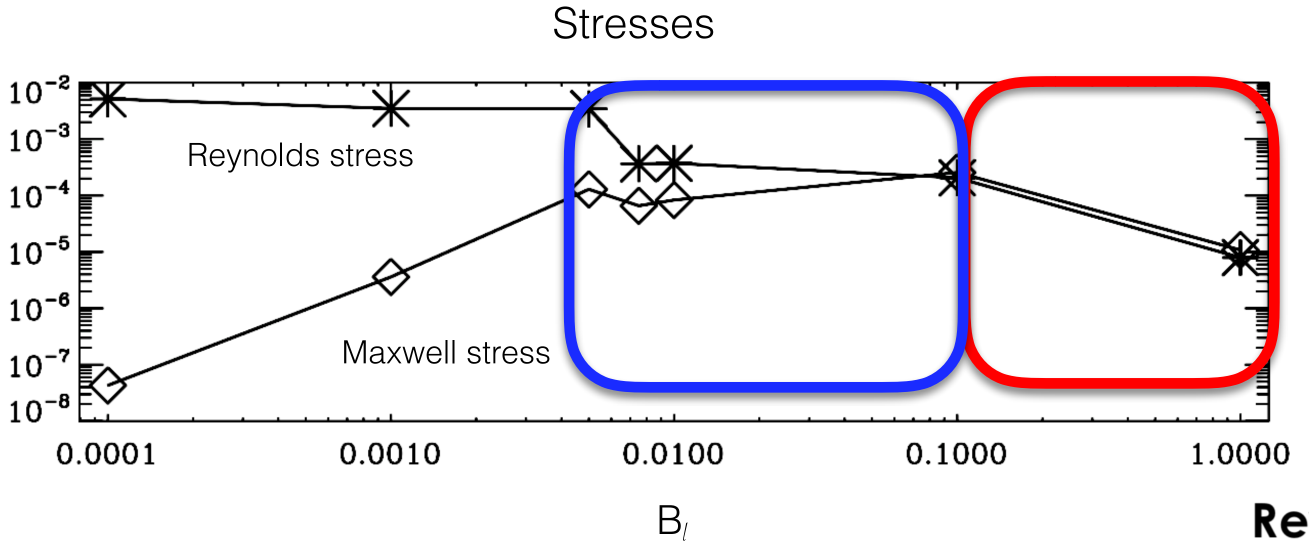
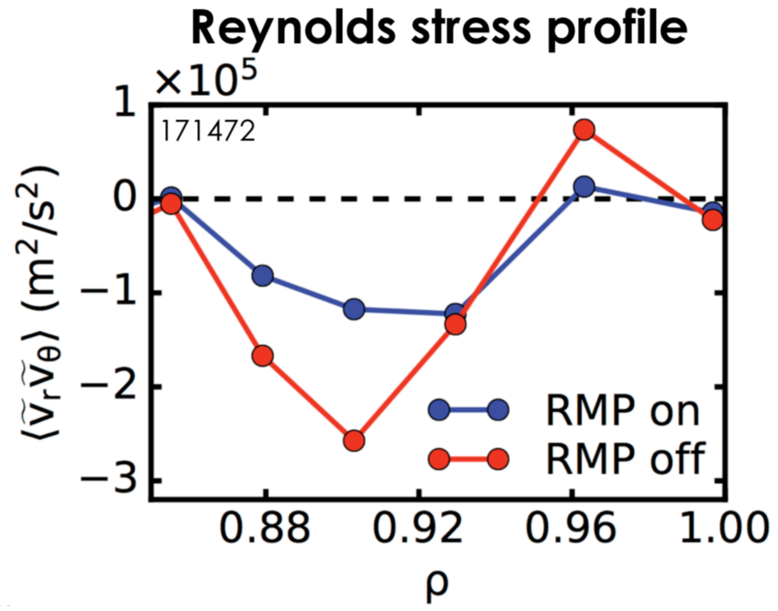


- **We obtained:**

1. **Dimensionless parameters** which successfully predict the transition line.
2. A theory of how the small-scale static **random magnetic fields suppress** the Reynolds stress.
3. This suppression happens when mean magnetic intensities **BELOW** that of Alfvènization.



(D. M. Kriete et al., TTF 2019)

Thank you!

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