# Cosmic Ray Diffusion with Magnetic Focusing

## Telegraph equation?

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# Abstract

The cosmic ray transport along the mean magnetic field, controlled by fast pitch-angle scattering on magnetic fluctuations, has been long considered to be largely diffusive. On the other hand, long-scale variations in the mean field add a convective component to the transport due to the magnetic mirror force.
The Chapman-Enskog approach is applied to the pitch-angle averaged spatial transport. Its convective part is shown to arise only due to this magnetic focusing effect. No “telegrapher” (mirror force independent) term emerges in any order of expansion, contrary to claims in the literature.

- spatial transport of CRs is fundamental to astrophysical problems (CR origin, likely to be accelerated in SNR shocks, solar energetic particle events, CR propagation through the galaxy or interplanetary space, etc.)[?, ?]
- much improved instruments probe new CR transport phenomena (hot spots in CR arrival direction, anomalies in rigidity spectra)
- CR transport equation, averaged in pitch angle, is often claimed to become hyperbolic (telegrapher's equation)
- canonical Chapman-Enskog transport treatment does not result in telegraph equation
- no time derivative higher than the first order one emerges in the pitch-angle averaged transport equation

# Equations

Energetic particles (e.g., CRs) in a magnetic field that varies smoothly on the particle gyro-scale are transported according to the drift-kinetic equation

$$\frac{\partial f}{\partial t}+v\mu\frac{\partial f}{\partial z}+v\sigma\left(1-\mu^2\right)\frac{\partial f}{\partial \mu}=\frac{\partial}{\partial \mu}\nu D\left(\mu\right)\left(1-\mu^2\right)\frac{\partial f}{\partial \mu} \quad (1)$$

$v$ and $\mu$ are the particle velocity and pitch angle
$z$ points in the local field direction
$\sigma=-B^{-1}\partial B/\partial z$ is the magnetic mirror inverse scale
$\nu$ is the (constant) pitch angle scattering rate, $D\left(\mu\right)\sim 1$

- the fastest transport is in $\mu$:
- small parameter of the treatment

$$\varepsilon=\frac{v}{l\nu}\ll 1$$

$l$ is the characteristic scale of the problem, e.g. the scale of $B\left(z\right)$ variation, CR source scale etc.

# Equations/Analysis

Measuring time in $\nu^{-1}$ and $z$ in $l$, the equation rewrites

$$\frac{\partial f}{\partial t} - \frac{\partial}{\partial \mu} D(\mu)\left(1-\mu^2\right)\frac{\partial f}{\partial \mu} = -\varepsilon\left(\mu\frac{\partial f}{\partial z} + \frac{\sigma}{2}\left(1-\mu^2\right)\frac{\partial f}{\partial \mu}\right) \quad (2)$$

- asymptotic expansion in $\varepsilon \ll 1$ is due to Chapman, Enskog, Hilbert
- expand $f$ in a series

$$f = f_0 + \varepsilon f_1 + \varepsilon^2 f_2 + \cdots \equiv f_0 + \tilde{f} \quad (3)$$

where

$$\langle f \rangle = f_0, \quad \text{with} \quad \langle \cdot \rangle = \frac{1}{2}\int_{-1}^{1} (\cdot)\, d\mu \quad (4)$$

that is $\langle \tilde{f} \rangle = 0$

# Analysis/Telegraph Equation

- equation for $f_0$, ("master equation")

$$\frac{\partial f_0}{\partial t} = -\varepsilon \frac{\partial}{\partial z}(1+\sigma)\langle \mu f \rangle = \frac{\varepsilon^2}{2}\left(\frac{\partial}{\partial z}+\sigma\right)\sum_{n=1}^{\infty}\varepsilon^{n-1}\left\langle \left(1-\mu^2\right)\frac{\partial f_n}{\partial \mu}\right\rangle \quad (5)$$

- as in Lorenz's gas $f_0$ depends on a "slow time" $t_2 = \varepsilon^2 t$ rather than on $t$

This may suggest to attribute the time derivative term in eq.(2) to the higher order of approximation (thus moving it to the rhs). Such ordering has been employed in e.g., [?] (see also refs to preceding works). A series of terms with progressively higher time derivatives will be generated on the rhs of eq.(5). From this series a term $\propto \partial^2 f_0/\partial t^2$ has been retained which formally converts the convective-diffusive eq.(5) into a "telegrapher's" equation

$$\frac{\partial f_0}{\partial t} + \tau \frac{\partial^2 f_0}{\partial t^2} = \kappa_{\parallel}\frac{\partial^2 f_0}{\partial z^2} + \frac{\kappa_{\parallel}}{l}\frac{\partial f_0}{\partial z}$$

# Analysis cont'd

- from equation (2) $\rightarrow$
- $\tilde{f}$ does depend on $t$ (as on the "fast" time)
- it is thus illegitimate to attribute the first term on the lhs to any order of approximation higher than the second term, notwithstanding this fast time dependence dies out for $t \gtrsim 1$.
- using eq.(2) we thus have

$$\frac{\partial f_n}{\partial t} - \frac{\partial}{\partial \mu} D(\mu)\left(1-\mu^2\right)\frac{\partial f_n}{\partial \mu} = -\mu \frac{\partial f_{n-1}}{\partial z} - \frac{\sigma}{2}\left(1-\mu^2\right)\frac{\partial f_{n-1}}{\partial \mu} \quad (6)$$
$$\equiv \Phi_{n-1}(t,\mu,z)$$

However, this form of equation is still not suitable for building an asymptotic series $f_n$ to submit to the master eq.(5). Indeed, the solubility condition for $f_2$ is
$\langle \Phi_1 \rangle = \frac{1}{2}(\partial/\partial z + \sigma)\left\langle \left(1-\mu^2\right)/D \right\rangle \partial f_0/\partial z = 0$
- too strong restriction suggestive of a wrong ordering.

# Multi-time expansion

- $f_0$ depends on the slow time $t_2$ so that *multiple time scales* are involved in the problem
- multi-time asymptotic expansion: Chapman-Enskog asymptotic method the operator $\partial/\partial t$ is expanded instead
- equivalently, introduce a hierarchy of independent variables $t \to t_0, t_1, \ldots$

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial t_0} + \varepsilon \frac{\partial}{\partial t_1} + \varepsilon^2 \frac{\partial}{\partial t_2} \ldots \tag{7}$$

Eq.(6) takes the form

$$\frac{\partial f_n}{\partial t_0} - \frac{\partial}{\partial \mu} D(\mu)\left(1-\mu^2\right)\frac{\partial f_n}{\partial \mu} = -\mu \frac{\partial f_{n-1}}{\partial z} - \frac{\sigma}{2}\left(1-\mu^2\right)\frac{\partial f_{n-1}}{\partial \mu} \tag{8}$$

$$-\sum_{k=1}^{n} \frac{\partial f_{n-k}}{\partial t_k} \equiv \mathscr{L}_{n-1}[f](t_0, \ldots, t_n; \mu, z)$$

Solution can be written as

# Multi-time Expansion Cont'd

$$\frac{\partial \tilde{f}_n}{\partial t_0} - \frac{\partial}{\partial \mu} D(\mu)\left(1-\mu^2\right)\frac{\partial \tilde{f}_n}{\partial \mu} = \mathscr{L}_{n-1}\left[\tilde{f}\right](t_0,\ldots,t_n;\mu,z) \qquad (10)$$

and

$$-\frac{\partial}{\partial \mu} D(\mu)\left(1-\mu^2\right)\frac{\partial \bar{f}_n}{\partial \mu} = \mathscr{L}_{n-1}\left[\bar{f}\right](t_2,\ldots,t_n;\mu,z) \qquad (11)$$

The solutions can be easily found for any $n$ using the eigenfunctions of the diffusion operator on the l.h.s. of the last equation

$$-\frac{\partial}{\partial \mu} D(\mu)\left(1-\mu^2\right)\frac{\partial \psi_k}{\partial \mu} = \lambda_k \psi_k$$

For $D=1$, for example, $\psi_k$ are the Legendre polynomials with $\lambda_k = k(k+1)$, $k = 0, 1, \ldots$.

# Asymptotic Expansion Cont'd

from eq.(8) we have

$$\frac{\partial f_0}{\partial t_0} = 0 \tag{12}$$

solubility condition for $f_1$ (obtained by integrating both sides of eq.[8] in $\mu$)

$$\frac{\partial f_0}{\partial t_1} = 0. \tag{13}$$

The solution for $f_1$ takes the form

$$\tilde{f}_1 = \sum_{k=1}^{\infty} C_k e^{-\lambda_k t_0} \psi_k(\mu) \tag{14}$$

We used the fact that the r.h.s. of eq.(10) vanishes for $n = 1$

# Asymptotic expansion cont'd

Furthermore,

$$\bar{f}_1 = \frac{1}{2}\left[\left\langle\frac{1-\mu}{D}\right\rangle - \int_{-1}^{\mu}\frac{d\mu}{D}\right]\frac{\partial f_0}{\partial z} \tag{15}$$

constants $C_k$ are determined by the initial condition for $\tilde{f}_1$
The solubility condition for $f_2$ yields a nontrivial result

$$\frac{\partial f_0}{\partial t_2} = -\frac{1}{2}\left(\frac{\partial}{\partial z}+\sigma\right)\left\langle\left(1-\mu^2\right)\frac{\partial f_1}{\partial \mu}\right\rangle. \tag{16}$$

The solubility condition for $f_3$ is

$$\frac{\partial f_0}{\partial t_3} = \underbrace{-\frac{1}{2}\left(\frac{\partial}{\partial z}+\sigma\right)\left\langle\left(1-\mu^2\right)\frac{\partial f_2}{\partial \mu}\right\rangle}_{\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots} \tag{17}$$

cf. eq.[5]

# Results

- the process may be continued *ad infinitum* since terms containing $\langle (1-\mu^2)\,\partial f_n/\partial\mu \rangle$ can be expressed through $f_{n-1}$, $f_{n-2}, \ldots$ down to $f_1$ given by eqs.(14) and (15).
- of interest is the evolution of $f_0$ on the time scales $t \gtrsim \varepsilon^{-2}$, so
- neglect contributions $\tilde{f}_n$ in the solubility conditions (as they become exponentially small) and retain only $\bar{f}_n$- parts.
- Multiplying eq.(13) by $\varepsilon$, eq.(16) by $\varepsilon^2$, eq.(17) by $\varepsilon^3$ and so on, summing up all the equations starting from eq.(12) we obtain on the l.h.s. of the new equation simply $\partial f_0/\partial t$ (see eq.[7]). Thus, no time derivatives of the order higher than the first one emerges.

# Master Equation to $\varepsilon^3$

- *evolution* equation for $f_0(t,z)$ up to the third order in $\varepsilon$ takes the following form

$$\frac{\partial f_0}{\partial t} = \frac{\varepsilon^2}{4}\left(\frac{\partial}{\partial z}+\sigma\right)\left\{\left\langle\frac{1-\mu^2}{D}\right\rangle + \right. \tag{18}$$

$$\left. +\frac{\varepsilon}{2}\left(\frac{\partial}{\partial z}+\sigma\right)\left[\left\langle\frac{1}{D}\int_{-1}^{\mu}\frac{1-\mu^2}{D}d\mu\right\rangle - \left\langle\frac{1+\mu}{D}\right\rangle\left\langle\frac{1-\mu^2}{D}\right\rangle\right]\right\}\frac{\partial f_0}{\partial z}$$

- first line: conventional diffusion and magnetic focusing
- second line: higher order transport due to asymmetric $D(\mu)$ (vanishes for even $D$)

# Summary

- multi-time asymptotic expansion is applied to CR transport with magnetic focusing
- no telegraph-type terms are recovered
- transport equation retains it's evolutionary character after pitch-angle averaging

- Outlook
  - proof using exact solutions in simplified cases
  - applications to CR modified shocks (B changes gradually)
  - applications to CR transport in Heliosphere

# For Further Reading I

C. Cercignani, Boltzmann Equation
*Applied Mathematical Sciences Series*
Springer-Verlag, 1988.

Y. E. Litvinenko and R. Schlickeiser
Telegraph equation for cosmic-ray transport
*A&A 554, A59 (2013)*