Wave Turbulence, Zonal Jets and Staircase Patterns in Fluids and Plasmas

P.H. Diamond

U. C. San Diego

Physics Colloquium: IIT, Delhi
March 22, 2021

This research was supported by the U.S. Department of Energy, Office of Science, Office of Fusion Energy Sciences, under Award Number DEFG02-04ER54738.

Outline

- Primer in Confinement: Pipes and Donuts

  The problem of scales

- Scale Selection I
  - Shear flow effects
  - Constructing models → potential vorticity (GFD and Plasma)
  - Physics of zonal flows
  - Closing the feedback loop: predator-prey analogue
- Scale Selection II – the ExB staircase
  - Staircases
  - Model-Bistable mixing
  - Some results
- Open Issues and Current Research

Magnetically confined plasma → tokamaks

- Nuclear fusion: option for generating large amounts of carbon-free energy – “30 years in the future and always will be… “
- Challenge: ignition -- reaction release more energy than the input energy

Lawson criterion:

$$n_i \tau_E T_i > 3 \times 10^{21} \mathrm{m}^{-3} \mathrm{s\,keV}$$

$\uparrow$

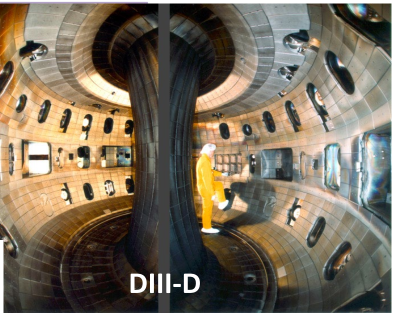
→ confinement $\qquad \tau_E \sim \frac{W}{P_{in}}$

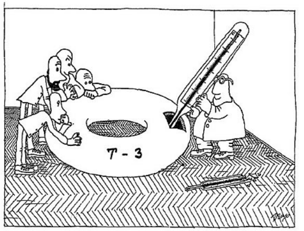
→ turbulent transport

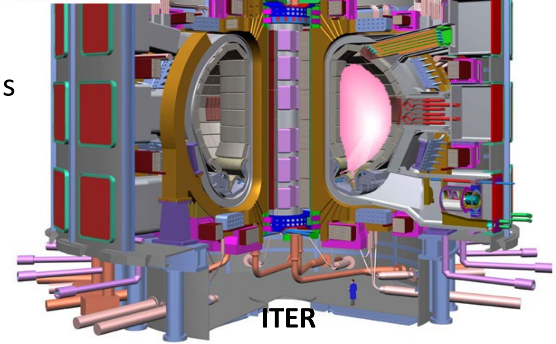
- Turbulence: instabilities and collective oscillations

→ low frequency modes dominate the transport ($\omega < \Omega_{ci}$)

- Key problem: Confinement, especially scaling
- NB: Not the only problem





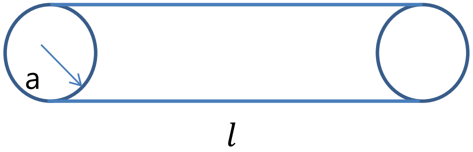


A Simpler Problem:

→ Drag in Turbulent Pipe Flow

- Essence of confinement problem:
  - given device, sources; what profile is achieved?
  - $\tau_E = W/P_{\text{in}}$, How optimize W, stored energy
- Related problem: Pipe flow → drag ↔ momentum flux

ΔP → pressure drop



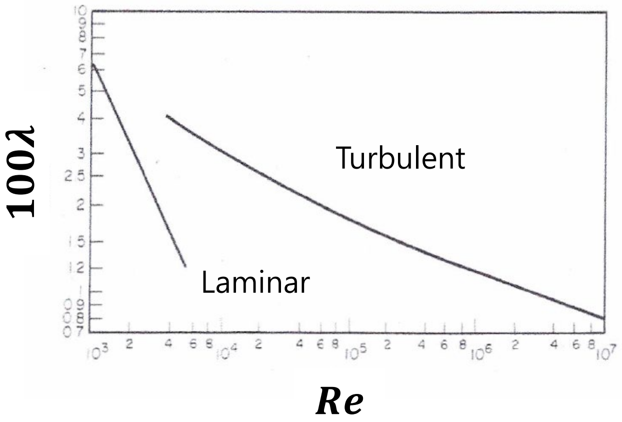
$$\Delta P \pi a^2 = \rho V_*^2 2\pi a l$$

→ friction velocity V_* ↔ u

Balance: momentum transport to wall

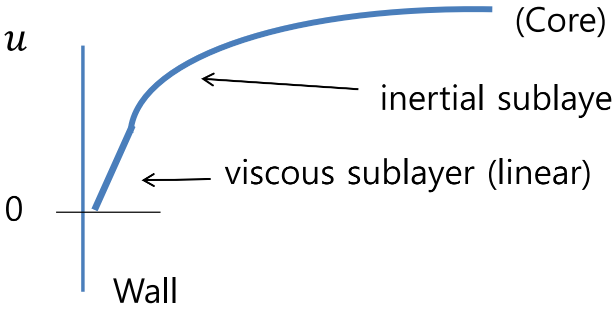
(Reynolds stress) vs ΔP

➔ Flow velocity profile



$$\lambda = \frac{2a\Delta P/l}{(1/2)\rho u^2}$$

Friction factor



inertial sublayer → ~ logarithmic (~ universal)

viscous sublayer (linear)

- Problem: physics of ~ universal logarithmic profile?
- Universality → scale invariance

- Prandtl Mixing Length Theory (1932)
  - Wall stress = $\rho V_*^2 = -\rho \nu_T \, \partial u / \partial x$ or: $\dfrac{\partial u}{\partial x} \sim \dfrac{V_*}{x}$ ← Spatial counterpart of K41

    eddy viscosity

    Scale of velocity gradient?
  - Absence of characteristic scale →

$$\nu_T \sim V_* x$$

$$u \sim V_* \ln(x/x_0)$$

$x \equiv$ mixing length, distance from wall

Analogy with kinetic theory …

$\nu_T = \nu \rightarrow x_0$, viscous layer → $x_0 = \nu / V_*$

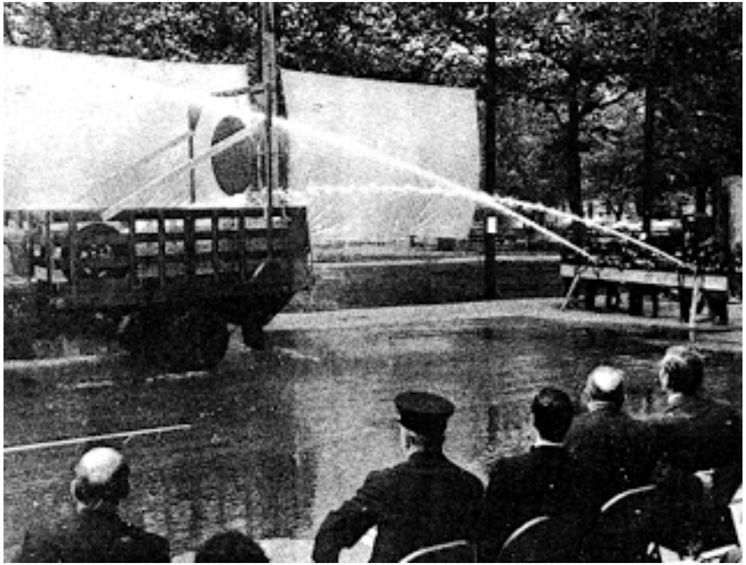
Some key elements:

- Momentum flux driven process ↔ contrast fixed profile
- Turbulent diffusion model of transport - eddy viscosity
- Mixing length – scale selection

  ~ x → macroscopic, eddys span system $x_0 < x < a$

  → ~ flat profile – strong mixing
- Self-similarity → x ↔ no scale, within $[x_0, a]$
- Reduce drag by creation of buffer layer i.e. steeper gradient than inertial sublayer (due polymer) – enhanced momentum confinement

  [N.B. : Analogue of H-mode]



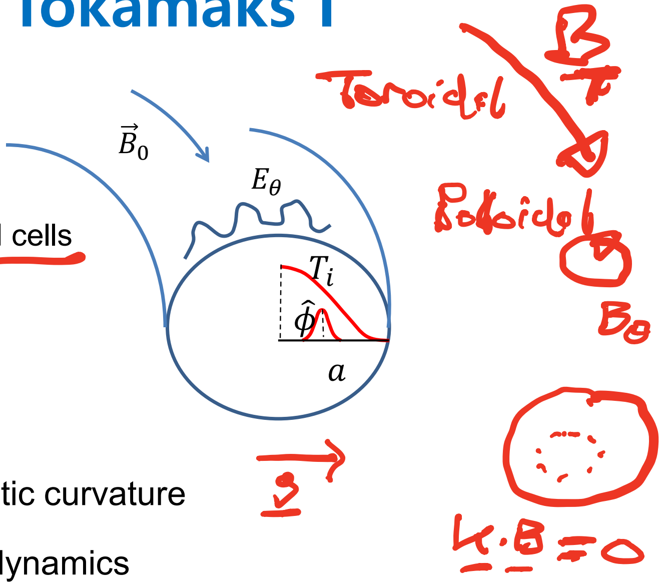
Without vs With Polymers Comparison → NYFD 1969

Turbulence in Tokamaks

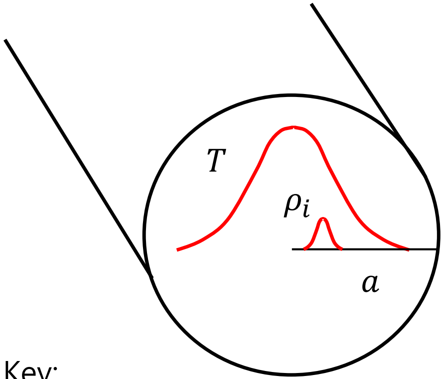
→ A Primer

Primer on Turbulence in Tokamaks I

- Strongly magnetized
  - Quasi 2D cells, Low Rossby #
  - * Localized by $\vec{k}\cdot\vec{B}=0$ (resonance) – pinned cells
- $\vec{V}_\perp = +\frac{c}{B}\vec{E}\times\hat{z}$, $\frac{V_\perp}{l\Omega_{ci}} \sim R_0 \ll 1$
- $\nabla T_e, \nabla T_i, \nabla n$ driven
- Akin to thermal convection with: g → magnetic curvature
- → $\mathrm{Re} \approx VL/\nu$ ill defined, not representative of dynamics
- Resembles ‘wave turbulence’, not high Re Navier-Stokes turbulence
- → $K \sim \tilde{V}\tau_c/\Delta \sim 1$ → *Kubo #* ≈ 1
- → Broad dynamic range, due electron and ion scales, i.e. a, ρ_i, ρ_e



Primer on Turbulence in Tokamaks II



Key:

2 scales:

$\rho \equiv$ gyro-radius

$a \equiv$ cross-section

$\rho_* \equiv \rho/a$ ➔ key ratio

$\rho_* \ll 1$

- Characteristic scale ~ few ρ_i → "mixing length"
- Characteristic velocity $v_d \sim \rho_* c_s$

- Transport scaling: $D_{GB} \sim \rho\, V_d \sim \rho_*\, D_B$ - Gyro-Bohm

  $D_B \sim \rho\, c_s \sim T/B$ - Bohm

- i.e. Bigger is better! ➔ sets profile scale via heat balance (Why ITER is huge…)
- Reality: $D \sim \rho_*^{\alpha}\, D_B$, $\alpha < 1$ ➔ 'Gyro-Bohm breaking'
- 2 Scales, $\rho_* \ll 1$ ➔ key contrast to pipe flow

THE Question ↔ Scale Selection

- Expectation (from pipe flow):
  - $l \sim a$
  - $D \sim D_B$
- Hope (mode scales)
  - $l \sim \rho_i$
  - $D \sim D_{GB} \sim \rho_* D_B$
- Reality: $D \sim \rho_*^{\alpha} D_B, \quad \alpha < 1$

Why? What physics competition sets α?

Scale Selection

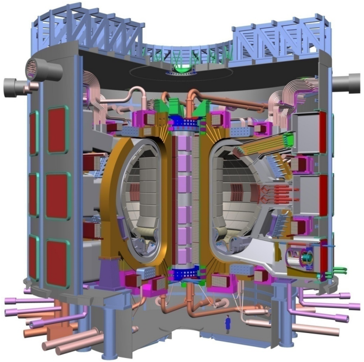
Zonal Flow Jets Natural to Planets, Tokamaks

- Zonal Flows Ubiquitous for: $R_0 \equiv$ Rossby # $= \tilde{V} / LR < 1$

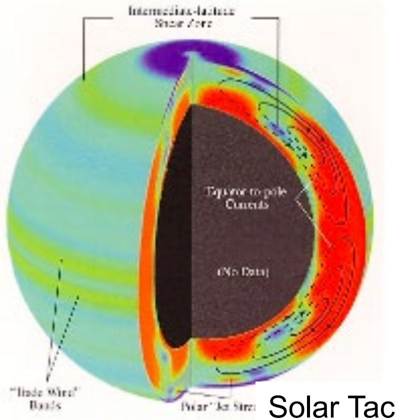
  ~ 2D fluids / plasmas $R_0 < 1$

  Rotation $\vec{\Omega}$, Magnetization $\vec{B}_0$, Stratification

  Ex: MFE devices, giant planets, stars…



➔ Sheared Flow



Solar Tachocline

Shear Flows – Significance?

No shear

shear

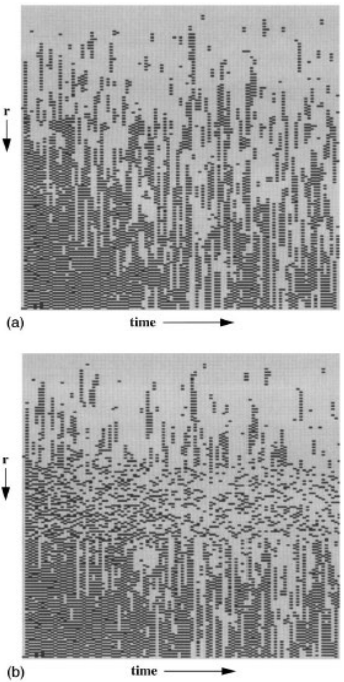


FIG. 11. Time evolution of the overturning sites (like Fig. 4). The avalanches do not appear continous in time because only every 50th time step is shown. (a) The shear-free case shows avalanches of all lengths over the entire radius. (b) The case with sheared flow shows the coherent avalanches being decorrelated in the shear zone in the middle of the pile.

How is transport affected by shear flows ?

➔ shear decorrelation!

Back to sandpile model:

2D pile + sheared flow of grains

Shearing flow decorrelates Toppling sequence

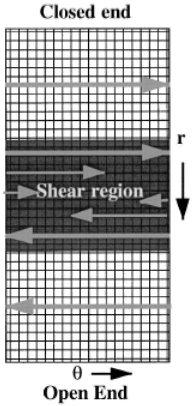
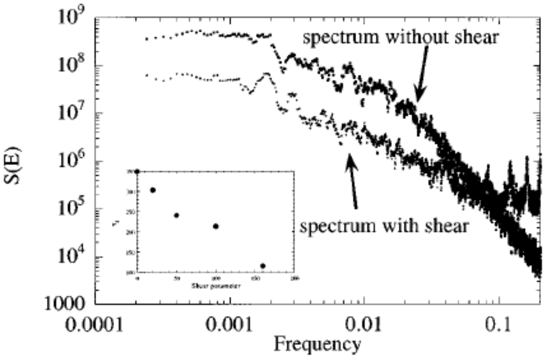


FIG. 10. A cartoon of the sandpile with a shear flow zone. The whole pile is flowing to the right at the top and to the left at the bottom connected by a variable sized region of sheared flow.

Avalanche coherence destroyed by shear flow

- Implications:

Spectrum of Avalanches



N.B.

- Profile steepens for unchanged toppling rules
- Distribution of avalanches changed

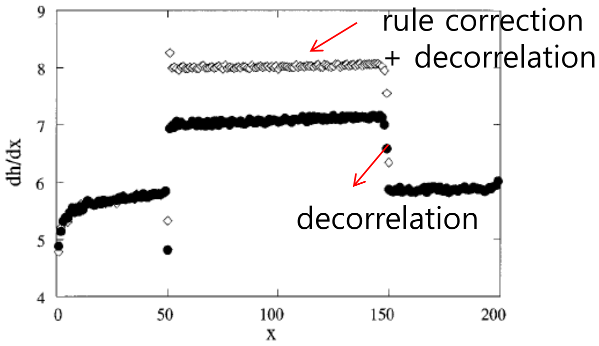


FIG. 14. The slopes of a sandpile with a shear region in the middle, including all the shear effects (diamonds) and just the transport decorrelation and the linear effect (circles).

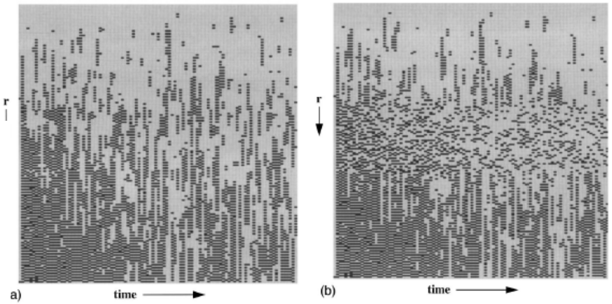


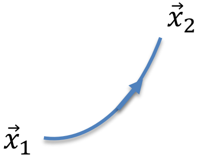
FIG. 11. Time evolution of the overturning sites (like Fig. 4). The avalanches do not appear continous in time because only every 50th time step is shown. (a) The shear-free case shows avalanches of all lengths over the entire radius. (b) The case with sheared flow shows the coherent avalanches being decorrelated in the shear zone in the middle of the pile.

How, Why Do Flows Form?

→ Models and Potential Vorticity

Flows

- GFD → The Fluid Dynamics of <u>Potential Vorticity</u> (R. Salmon)
- Ditto for confined plasmas... (PD)
- What is PV ?
  - Consider freezing-in:



$$\frac{d\vec{l}}{dt} = \vec{V}(\vec{x}_2) - \vec{V}(\vec{x}_1) = \vec{l} \cdot \nabla\vec{V}$$

[$\vec{l}$ “frozen in” to flow V]

  - Flow ↔ Vorticity $\vec{\omega} = \nabla \times \vec{V}$

$$\frac{\partial\vec{V}}{\partial t} + \vec{V} \cdot \nabla\vec{V} = -\frac{\nabla P}{\rho} - 2\vec{\Omega} \times \vec{V}$$

Coriolis / Lorentz

PV cont’d

- Then, for $P = P(\rho)$:

$$\partial_t(\vec{\omega} + 2\vec{\Omega}) = \nabla \times [\vec{V} \times (\vec{\omega} + 2\vec{\Omega})]$$

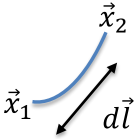
$$\frac{d}{dt}\left(\frac{\vec{\omega}+2\vec{\Omega}}{\rho}\right) = \left(\frac{\vec{\omega}+2\vec{\Omega}}{\rho}\right) \cdot \nabla\vec{V}$$

$$\frac{d}{dt}\vec{l} = \vec{l} \cdot \nabla\vec{V} \quad \rightarrow \quad \frac{\vec{\omega}+2\vec{\Omega}}{\rho}$$ “frozen in”

→ non-trivial, frozen in ≠ passive !

- Passive Scalar ψ

$$\frac{d}{dt}\psi = 0$$



$$\Rightarrow \frac{d}{dt}\delta\psi = 0 \;; \qquad \delta\psi = \nabla\psi \cdot d\vec{l}$$

PV cont'd

$$\frac{d}{dt}\left(\nabla\psi \cdot d\vec{l}\right) = 0$$

but $\frac{d}{dt} d\vec{l} = d\vec{l} \cdot \nabla\vec{V}$

$$\updownarrow$$

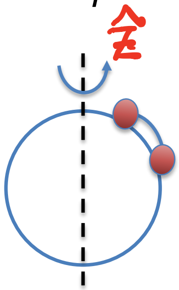
$$\frac{\mathrm{d}}{\mathrm{dt}}\left(\frac{\vec{\omega}+2\vec{\Omega}}{\rho}\right) = \left(\frac{\vec{\omega}+2\vec{\Omega}}{\rho}\right) \cdot \nabla\vec{V}\ ,$$ so

- $\boxed{\frac{d}{dt}\left[\left(\vec{\omega} + 2\vec{\Omega}\right) \cdot \frac{\nabla\psi}{\rho}\right] = 0}$ statement of <u>PV conservation</u>
- $PV = \underset{\uparrow}{q} = \frac{(\vec{\omega}+2\vec{\Omega})}{\rho} \cdot \nabla\psi$ { Good analogy with conserved charge }

PV Conservation ↔ Trade-offs

$$q = \frac{(\vec{\omega} + 2\vec{\Omega})}{\rho} \cdot \nabla\psi$$

- displace latitude → $\vec{\omega}$ changes



- displace in $\left\{ \begin{array}{l} \text{density} \\ \text{thickness} \end{array} \right.$ → $\vec{\omega}$ changes

etc.

PV cont'd

- Conservation ↔ Symmetry ? - ala' Noether

Particle re-labeling; $\vec{x}(s,\tau)$ $s \rightarrow s' = s + \delta s$

[PV conserved when particles can be re-labeled, without changing the thermodynamic state]

- Related: Kelvin's Theorem ($\rho =$ const. $P = P(\rho)$)

$$\frac{d}{dt}\left[\int d\vec{a} \cdot \left(\vec{\omega} + 2\vec{\Omega}\right)\right] = 0$$

→ total circulation conserved (parcel + planetary)

From Kelvin's Theorem to the β Plane Model (Charney)

- Kelvin's Theorem for rotating system

$$\omega \to \omega + 2\mathbf{\Omega}$$

relative planetary

$$\longrightarrow \quad \oint \mathbf{v}\cdot d\mathbf{l} = \int d\mathbf{a}\cdot(\omega + 2\Omega) \equiv C$$

$$\dot{C} = 0$$

- $Ro = V/(2\Omega L) \ll 1 \quad \to \quad \mathbf{V} \cong -\nabla_{\perp} p \times \hat{z}/(2\Omega)$ geostrophic balance

$\to$ 2D dynamics

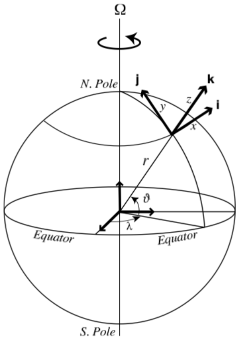
- Displacement on beta plane

$$\dot{C} = 0 \quad \to \quad \frac{d}{dt}\omega \cong -\frac{2\Omega}{A}\sin\theta_0 \frac{dA}{dt}$$

$$= -2\Omega\frac{d\theta}{dt} = -\beta V_y$$

$$\omega = \nabla^2\phi \qquad \beta = 2\Omega\sin\theta_0/R$$

So...



Charney Equation, cont'd

- Charney equation $\frac{d}{dt}(\omega + \beta y) = 0$

- Locally Conserved PV $q = \omega + \beta y$

parcel ↗ ↖ planetary

n.b. topography

$$q = \omega / H + \beta y$$

- Latitudinal displacement → change in relative vorticity

- Linear consequence → Rossby Wave

$$\omega = -\beta k_x / k^2$$

$\omega = 0$ → zonal flow

$k_x = 0$ → azimuthal symmetry

observe: $v_{g,y} = 2\beta \underline{k_x k_y} / (k^2)^2$

→ Rossby wave intimately connected to momentum transport

➔ Reynolds stress $\langle V_x V_y \rangle$

- Latitudinal PV Flux → circulation

PV Dynamics – Plasmas ?

- Isn't this about plasmas, too?

- $q = (\vec{\omega} + 2\vec{\Omega}) \cdot \frac{\nabla\psi}{\rho}$

$$2\vec{\Omega} \rightarrow \Omega_i \hat{z}$$

now $\rho \rightarrow n_0(r) + \tilde{n}$

$$\vec{\nabla}\psi \rightarrow \hat{z}$$

So $\frac{d}{dt}\left[\frac{\omega_z + \Omega_i}{n_0(r) + \tilde{n}}\right] = 0$

$$\Rightarrow \frac{d}{dt}\tilde{\omega}_z - \Omega_i \frac{1}{n_0} \frac{\widetilde{dn_i}}{dt} = 0$$

ala' Geostrophic:

$$\begin{cases} \vec{V} = -\frac{c}{B}\nabla\phi \times \hat{z} \\ \omega_z = \frac{c}{B_0}\nabla^2\phi \end{cases}$$

with $V_{thi} \ll \frac{\omega}{k_\parallel} < V_{the}$ $\quad \frac{\tilde{n}_i}{n_0} \sim \frac{\tilde{n}_e}{n_0} \sim \frac{|e|\widehat{\phi}}{T}$

$$\rightarrow \boxed{\frac{d}{dt}\left(\frac{|e|\widehat{\phi}}{T} - \rho_s^2 \nabla_\perp^2 \frac{|e|\widehat{\phi}}{T}\right) + V_* \partial_y \frac{|e|\widehat{\phi}}{T} = 0}$$

Hasegawa-Mima Eqn.

→ PV conservation

PV and Models - Plasmas

- Hasegawa-Mima, prototype:

$$\frac{d}{dt}(\phi - \rho_s^2 \nabla^2 \phi + \ln n_0(r)) = 0$$

- tip of iceberg of zoology of systems
- captures essence
- in tokamak, zonal flows have: $k_{\parallel} = 0$ and $k_\theta = 0$

$$\frac{d}{dt} \nabla^2 \phi = 0$$

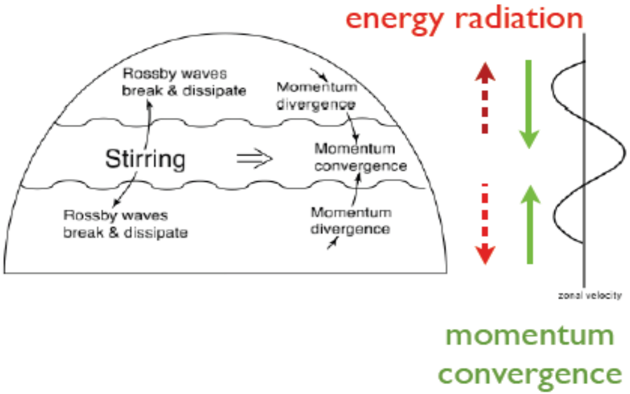
➔ generation of flow → $\langle \tilde{V}_r \nabla^2 \tilde{\phi} \rangle$ ➔ vorticity flux

Physics of Zonal Flows

→ How do Zonal Flow Form?

Simple Example: Zonally Averaged Mid-Latitude Circulation

- classic GFD example: Rossby waves + Zonal flow (c.f. Vallis '07, Held '01)
- Key Physics:



Rossby Wave:

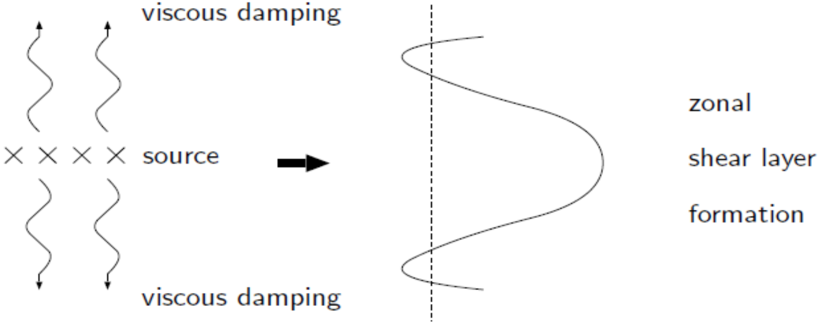
$$\omega_k = -\frac{\beta k_x}{k_\perp^2}$$

$$v_{gy} = 2\beta \frac{k_x k_y}{(k_\perp^2)^2}, \ \langle \tilde{v}_y \tilde{v}_x \rangle = \Sigma_{\vec{k}} -k_x k_y \left|\hat{\phi}_{\vec{k}}\right|^2$$

$\therefore v_{gy} v_{phy} < 0$ → Backward wave!

➔ Momentum convergence at stirring location

- …"the central result that a rapidly rotating flow, when stirred in a localized region, will converge angular momentum into this region." (I. Held, '01)
- Outgoing waves $\Rightarrow$ incoming wave momentum flux



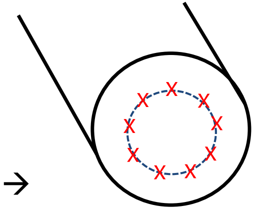
- Local Flow Direction (northern hemisphere):
  - eastward in source region
  - westward in sink region
  - set by $\beta > 0$
  - Some similarity to spinodal decomposition phenomena
    → Both 'negative diffusion' phenomena

←→ Cahn-Hillard Equation

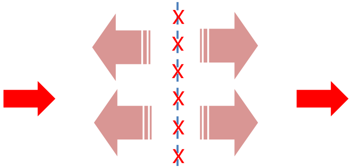
Wave-Flows in Plasmas

MFE perspective on Wave Transport in DW Turbulence

- localized source/instability drive intrinsic to drift wave structure



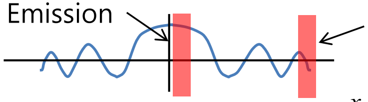
Fluctuations → Pinned to non-resonant surfaces



$x=0$

radial structure

– couple to damping ↔ outgoing wave



$$- \quad v_{gr} = -2\rho_s^2 \frac{k_\theta k_r v_*}{(1+k_\perp^2 \rho_s^2)^2}$$

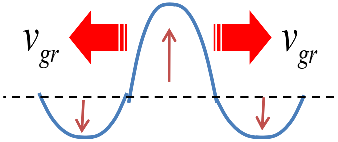
$$\langle v_{rE} v_{\theta E} \rangle = -\frac{c^2}{B^2} |\phi_{\vec{k}}|^2 k_r k_\theta < 0$$

$$x > 0 \quad \Rightarrow \quad v_{gr} > 0$$

$$x < 0 \quad \Rightarrow \quad v_{gr} < 0$$

$$v_* < 0 \quad \Rightarrow \quad k_r k_\theta > 0$$

- outgoing wave energy flux → incoming wave momentum flux
  → counter flow spin-up!



- zonal flow layers form at excitation regions

Plasma Zonal Flows I

- What is a Zonal Flow? – Description?
  - *n = 0* potential mode; *m = 0* (ZFZF), with possible sideband (GAM)
  - toroidally, poloidally symmetric *ExB* shear flow
- Why are Z.F.'s important?
  - Zonal flows are secondary (nonlinearly driven):
    - modes of minimal inertia (Hasegawa et. al.; Sagdeev, et. al. '78)
    - modes of minimal damping (Rosenbluth, Hinton '98)
    - drive zero transport (*n = 0*)
  - natural predators to feed off and retain energy released by gradient-driven microturbulence

i.e. ZF's soak up turbulence energy

Plasma Zonal Flows II

- Fundamental Idea:
  - Potential vorticity transport + 1 direction of translation symmetry
    → Zonal flow in magnetized plasma / QG fluid
  - Kelvin's theorem is ultimate foundation

- Charge Balance → polarization charge flux → Reynolds force
  - Polarization charge → $-\boxed{\rho^2}\nabla^2\phi = \boxed{n_{i,GC}(\phi) - n_e(\phi)}$

    polarization length scale *ion GC* *electron density*
  - so $\Gamma_{i,GC} \neq \Gamma_e$ → $\rho^2\left\langle \tilde{v}_{rE}\nabla_\perp^2\tilde{\phi}\right\rangle \neq 0$ ↔ 'PV transport'

    polarization flux → What sets cross-phase?
  - If 1 direction of symmetry (or near symmetry):

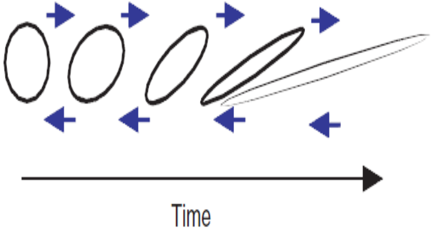
    $-\rho^2\left\langle \tilde{v}_{rE}\nabla_\perp^2\tilde{\phi}\right\rangle = -\partial_r\left\langle \tilde{v}_{rE}\tilde{v}_{\perp E}\right\rangle$ (Taylor, 1915)

    $-\partial_r\left\langle \tilde{v}_{rE}\tilde{v}_{\perp E}\right\rangle$ → Reynolds force → Flow   Recall $\langle\omega_Z\rangle$ evolution!

Zonal Flows Shear Eddys I

- Coherent shearing: (Kelvin, G.I. Taylor, Dupree'66, BDT'90)
  - radial scattering + $\langle V_E \rangle' \rightarrow$ hybrid decorrelation
  - $k_r^2 D_\perp \rightarrow (k_\theta^2 \langle V_E \rangle'^2 D_\perp / 3)^{1/3} = 1/\tau_c$

  ➔ <u>shearing restricts mixing scale!</u>



- Other shearing effects (linear):

Response shift and dispersion

  - spatial resonance dispersion: $\omega - k_{\|} v_{\|} \Rightarrow \omega - k_{\|} v_{\|} - k_\theta \langle V_E \rangle' (r - r_0)$
  - differential response rotation → especially for kinetic curvature effects

Quasi-Particle Model – Eddy Population Evolution

- Zonal Shears: Wave kinetics (Zakharov et. al.; P.D. et. al. ‘98, et. seq.)

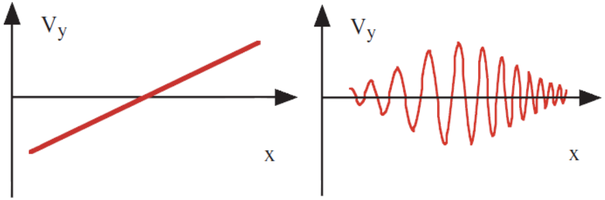
  Coherent interaction approach (L. Chen et. al.)

- $dk_r / dt = -\partial(\omega + k_\theta V_E) / \partial r \, ; \; V_E = \langle V_E \rangle + \widetilde{V}_E$

  Mean shearing $: k_r = k_r^{(0)} - k_\theta V_E' \tau$

  Zonal Random shearing $: \langle \delta k_r^2 \rangle = D_k \tau$

  $$D_k = \sum_q k_\theta^2 \left| \widetilde{V}'_{E,q} \right|^2 \tau_{k,q}$$



- Wave ray chaos (not shear RPA) underlies $D_k \rightarrow$ induced diffusion
- Induces wave packet dispersion
- Applicable to ZFs and GAMs

- Mean Field Wave Kinetics

$$\frac{\partial N}{\partial t} + (\vec{V}_{gr} + \vec{V}) \cdot \nabla N - \frac{\partial}{\partial r}(\omega + k_\theta V_E) \cdot \frac{\partial N}{\partial \vec{k}} = \gamma_{\vec{k}} N - C\{N\}$$

$$\Rightarrow \boxed{\frac{\partial}{\partial t}\langle N \rangle - \frac{\partial}{\partial k_r} D_k \frac{\partial}{\partial k_r} \langle N \rangle = \gamma_{\vec{k}} \langle N \rangle - \langle C\{N\} \rangle} \longleftarrow$$

Zonal shearing

→ Evolves population in response to shearing

Closing the Feedback Loop

→ Predator-Prey Analogue

Energetics

- Energetics: Books must Balance for Reynolds Stress-Driven Flows!
- Fluctuation Energy Evolution – Z.F. shearing

$$\int d\vec{k}\,\omega\left(\frac{\partial}{\partial t}\langle N\rangle - \frac{\partial}{\partial k_r} D_k \frac{\partial}{\partial k_r}\langle N\rangle\right) \Rightarrow \frac{\partial}{\partial t}\langle \varepsilon\rangle = -\int d\vec{k}\, V_{gr}(\vec{k}) D_{\vec{k}} \frac{\partial}{\partial k_r}\langle N\rangle \qquad V_{gr} = \frac{-2k_r k_\theta V_* \rho_s^2}{\left(1+k_\perp^2\rho_s^2\right)^2}$$

Point: For $d\langle\Omega\rangle / dk_r < 0$, Z.F. shearing damps wave energy

- Fate of the Energy: Reynolds work on Zonal Flow !

Modulational Instability

$$\partial_t \delta V_\theta + \partial\left(\delta\langle\tilde{V}_r\tilde{V}_\theta\rangle\right)/\partial r = \quad \gamma\delta V_\theta$$

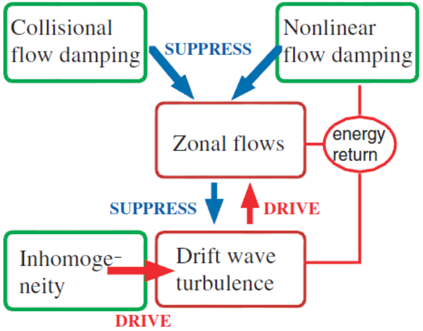
$$\delta\langle\tilde{V}_r\tilde{V}_\theta\rangle \sim \frac{k_r k_\theta \delta N}{(1+k_\perp^2\rho_s^2)^2}$$

N.B.: Wave decorrelation essential:
Equivalent to PV transport
(c.f. Gurcan et. al. 2010)

- Bottom Line:
  - Z.F. growth due to shearing of waves
  - “Reynolds work” and “flow shearing” as relabeling → books balance
  - Z.F. damping emerges as critical; MNR ‘97

Feedback Loops

- Closing the loop of shearing and Reynolds work
- Spectral 'Predator-Prey' Model



Collisional flow damping

SUPPRESS

Nonlinear flow damping

Zonal flows

energy return

SUPPRESS

DRIVE

Inhomogeneity

Drift wave turbulence

DRIVE

→ Self-regulating system → "ecology"

→ Mixing scale regulated

Prey → Drift waves, $\langle N \rangle$

$$\frac{\partial}{\partial t}\langle N\rangle - \frac{\partial}{\partial k_r} D_k \frac{\partial}{\partial k_r}\langle N\rangle = \gamma_k\langle N\rangle - \frac{\Delta\omega_k}{N_0}\langle N\rangle^2$$

Predator → Zonal flow, $|\phi_q|^2$

$$\frac{\partial}{\partial t}|\phi_q|^2 = \Gamma_q\left[\frac{\partial\langle N\rangle}{\partial k_r}\right]|\phi_q|^2 - \gamma_d|\phi_q|^2 - \gamma_{NL}[|\phi_q|^2]|\phi_q|^2$$

Feedback Loops II

- Recovering the 'dual cascade':
  - Prey → <N> ~ <Ω> ⇒ induced diffusion to high k_r $\left\{ \right.$ ⇒ Analogous → forward potential enstrophy cascade; PV transport
  - Predator → $|\phi_q|^2 \sim \left\langle V_{E,\theta}^2 \right\rangle$ $\left\{ \right.$
    - ⇒ growth of *n=0, m=0* Z.F. by turbulent Reynolds work
    - ⇒ Analogous → inverse energy cascade

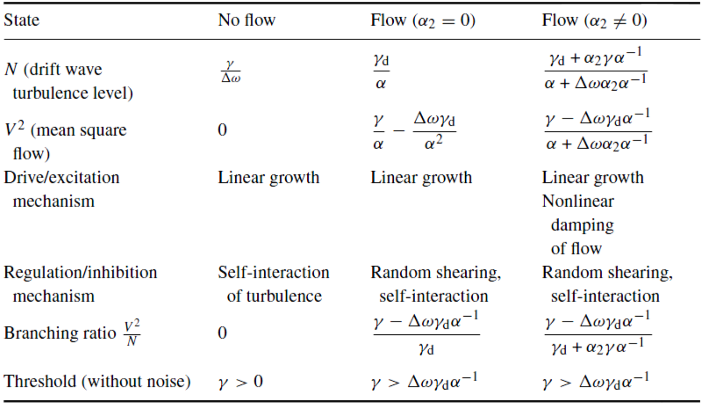
- Mean Field Predator-Prey Model

  (P.D. et. al. '94, DI²H '05)

$$\frac{\partial}{\partial t}N = \gamma N - \alpha V^2 N - \Delta\omega N^2$$

$$\frac{\partial}{\partial t}V^2 = \alpha N V^2 - \gamma_d V^2 - \gamma_{NL}(V^2)V^2$$

System Status



State	No flow	Flow ($\alpha_2 = 0$)	Flow ($\alpha_2 \neq 0$)
N (drift wave turbulence level)	$\frac{\gamma}{\Delta\omega}$	$\frac{\gamma_\mathrm{d}}{\alpha}$	$\frac{\gamma_\mathrm{d} + \alpha_2\gamma\alpha^{-1}}{\alpha + \Delta\omega\alpha_2\alpha^{-1}}$
V^2 (mean square flow)	0	$\frac{\gamma}{\alpha} - \frac{\Delta\omega\gamma_\mathrm{d}}{\alpha^2}$	$\frac{\gamma - \Delta\omega\gamma_\mathrm{d}\alpha^{-1}}{\alpha + \Delta\omega\alpha_2\alpha^{-1}}$
Drive/excitation mechanism	Linear growth	Linear growth	Linear growth Nonlinear damping of flow
Regulation/inhibition mechanism	Self-interaction of turbulence	Random shearing, self-interaction	Random shearing, self-interaction
Branching ratio $\frac{V^2}{N}$	0	$\frac{\gamma - \Delta\omega\gamma_\mathrm{d}\alpha^{-1}}{\gamma_\mathrm{d}}$	$\frac{\gamma - \Delta\omega\gamma_\mathrm{d}\alpha^{-1}}{\gamma_\mathrm{d} + \alpha_2\gamma\alpha^{-1}}$
Threshold (without noise)	$\gamma > 0$	$\gamma > \Delta\omega\gamma_\mathrm{d}\alpha^{-1}$	$\gamma > \Delta\omega\gamma_\mathrm{d}\alpha^{-1}$

Scale Selection II

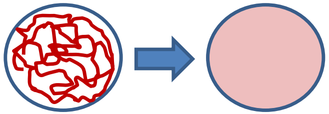
– the ExB Staircase

➔ Spatial Structure due Closed Feedback Loops

Dynamics in Real Space

- Conventional Wisdom → Homogenization ?!
  - Prandtl, Batchelor, Rhines:
  - PV homogenized:
    Shear + Diffusion

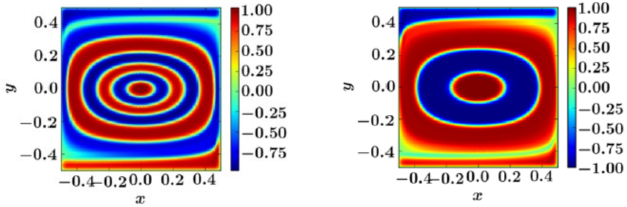
(2D fluid)



$\nabla q \to 0$

  - Mechanism: - Shear dispersion $\tau \sim \tau_{rot}\,(Re)^{1/3}$ → $\tau_{rot}\,Re$

    - Forward Enstrophy Cascade, ‘PV Mixing’
  - Introduce Bi-stable Mixing → Layers

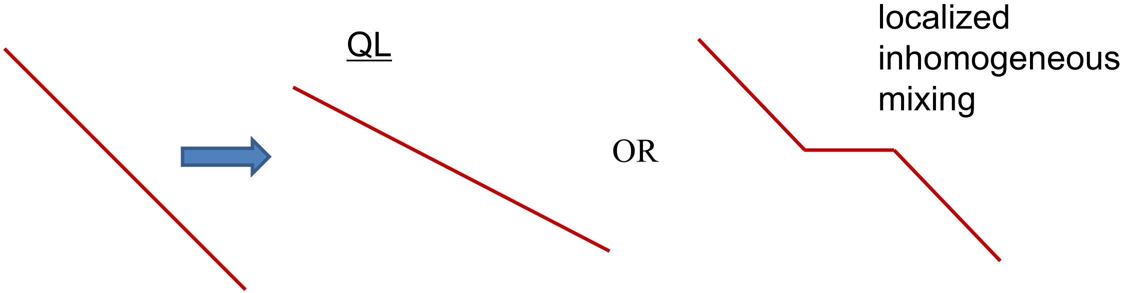


  - Cahn-Hilliard + Eddy Flow ←→ bistability

    → target pattern

(Fan, P.D., Chacon,
PRE Rap. Com. ‘17)

Fate of Gradient?

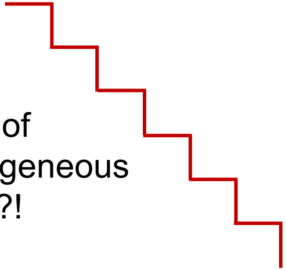


QL

OR

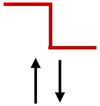
localized
inhomogeneous
mixing

OR - ‘staircase’



pattern of
inhomogeneous
mixing ?!

- layers, steps, corrugations
- shear layers ←→ relation to corrugations?



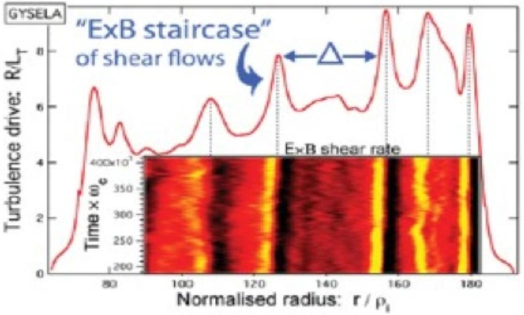
?

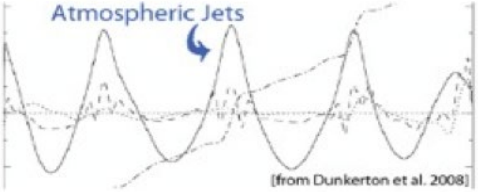
Zonal flows at corrugations ?

Spatial Structure: ExB staircase formation

- ExB flows often observed to self-organize structured pattern in magnetized plasmas
- \`ExB staircase’ is observed to form

(G. Dif-Pradalier, P.D. et al. Phys. Rev. E. ’10)





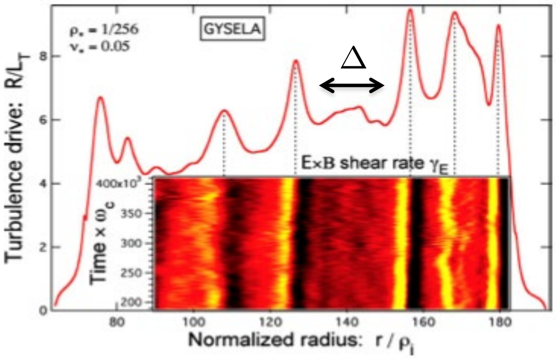
also: GK5D, Kyoto-Dalian-SWIP group, gKPSP, ... several GF codes

- flux driven, full f simulation
- Quasi-regular pattern of shear layers and profile corrugations (steps)
- Region of the extent $\Delta \gg \Delta_c$ interspersed by temp. corrugation/ExB jets

  → ExB staircases
- so-named after the analogy to PV staircases and atmospheric jets
- Step spacing → avalanche distribution outer-scale
- scale selection problem

ExB Staircase, cont'd

- Important feature: co-existence of shear flows and avalanches/spreading



- Seem mutually exclusive ?

  → strong ExB shear prohibits transport

  → mesoscale scattering smooths out corrugations

- Can co-exist by separating regions into:

  1. avalanches of the size $\Delta \gg \Delta_c$

  2. localized strong corrugations + jets

- How understand the formation of ExB staircase??

  - What is process of self-organization linking avalanche scale to ExB step scale?

  i.e. how explain the emergence of the step scale ?

- Some similarity to phase ordering in fluids – spinodal decomposition

Model → Bistable Mixing

Basic Equations ↔ Hasegawa-Wakatani (life beyond CHM)

$$\frac{d}{dt}\nabla_\perp^2\phi + \chi_{\parallel e}\nabla_\parallel^2(\phi - n) = \mu\nabla_\perp^2\nabla_\perp^2\phi$$

$$\frac{d}{dt}n + \chi_{\parallel e}\nabla_\parallel^2(\phi - n) = D_0\nabla_\perp^2 n$$

$$\frac{d}{dt} = \partial_t + \nabla\phi \times \hat{z}\cdot\nabla \qquad n = \langle n(x)\rangle + \tilde{n} \qquad \nabla_\perp^2\phi = \langle\nabla_\perp^2\phi(x)\rangle + \nabla_\perp^2\tilde{\phi}$$

- <u>PV</u> $q = n - \nabla_\perp^2\phi$ conserved! , to μ , D_0
- $\chi_\parallel \neq 0 \quad \rightarrow \quad \langle\tilde{v}_r\tilde{n}\rangle \neq 0$ ‘negative dissipation mechanism’ ➔ drift instability (Sagdeev, et. al.)

  $\omega \leq \omega_{*e} \quad \rightarrow \quad \langle\tilde{v}_r\tilde{n}\rangle > 0$

  shear
- ZF ➔ $k_\parallel = 0$ $\quad n \leftrightarrow \nabla_\perp^2\phi$ PV exchange
- ZF → $\langle\tilde{v}_r\nabla^2\tilde{\phi}\rangle$ → Reynolds force

  Corrugation → $\langle\tilde{v}\tilde{n}\rangle$ → particle flux

$\langle\tilde{n}\nabla^2\tilde{\phi}\rangle$?

c.f. singh, P.D. 2021

‘Bistable’ Mixing – A Simple Mechanism

- Mean field model with 2 mixing scales (after BLY 1998)
- So, for H-W:
- Density: $$\frac{\partial}{\partial t}\langle n\rangle = \frac{\partial}{\partial x}\left(D_n \frac{\partial\langle n\rangle}{\partial x}\right) + D_c \frac{\partial^2\langle n\rangle}{\partial x^2}$$
- Vorticity: $$\frac{\partial}{\partial t}\langle u\rangle = \frac{\partial}{\partial x}\left[(D_n - \chi)\frac{\partial\langle n\rangle}{\partial x}\right] + \chi\frac{\partial^2\langle u\rangle}{\partial x^2} + \mu_c \frac{\partial^2\langle u\rangle}{\partial x^2},$$

simple mixing + 2 length scale
→ staircase

- Enstrophy(intensity): $$\frac{\partial}{\partial t}\varepsilon = \frac{\partial}{\partial x}\left(D_\varepsilon \frac{\partial\varepsilon}{\partial x}\right) + \chi\left[\frac{\partial\langle n-u\rangle}{\partial x}\right]^2 - \varepsilon_c^{-1/2}\varepsilon^{3/2} + \gamma_\varepsilon \varepsilon.$$ → includes crude turbulence spreading model
- $D, \chi \sim \tilde{V} l_{mix}$

$$\boxed{l_{\text{mix}} = \frac{l_0}{\left(1 + l_0^2\left[\partial_x(n-u)\right]^2/\varepsilon\right)^{\kappa/2}},}$$

l_0 → excitation scale (drive)
l_R → Rhines scale (emergent)
ω_{MM} vs $\Delta\omega$ - can be generalized

↑
two scales!

- Scale cross-over → ‘transport bifurcation’
- $l_0/l_R < 1$ → strong mixing (eddys)
- $l_0/l_R > 1$ → weak mixing (waves) → sharpening feedback
- Is this ~ equivalent to ‘two-fluid’ mixing length model (E.A. Spiegel)

How, Why ?

- PV is mixed → natural for ‘mixing length model’ exploits conserved phase space density
- Potential Enstrophy is natural formulation – $\langle \delta f^2 \rangle$ for intensity → conserved
- Beyond BLY → 2 mean fields $\langle n \rangle$, $\langle \nabla^2 \phi \rangle$ + ε – fluctuation potential enstrophy

  → exchange and couplings
- Reynolds work and particle flux couple mean and fluctuations
- Nonlinear damping ↔ forward enstrophy cascade
- D_n, χ → turbulent transport coefficients are fundamental
- Glorified ‘$k - \epsilon$ model’

How, Why ? Cont'd

- $l_{mix} > \rho_s$ → simplifies inversion ($\nabla^2\phi \rightarrow V$)
- Dissipative DW ~ adiabatic regime: $k_{\parallel}^2 V_{the}^2/\nu \gg \omega$

  $D_n \approx \tilde{v}^2/\alpha \sim \epsilon l^2/\alpha \longrightarrow \langle v_r \tilde{n}\rangle$ phase fixed by α!

  Major simplification ⟶ <u>solid</u>, where applicable

  $\chi \sim D_n$ (non-resonant diffusion)
- $\langle \tilde{v}_r \nabla^2\phi\rangle = -\chi\partial_x\langle\nabla^2\phi\rangle + \Pi_{resid}[\nabla n]$

  $\langle\nabla^2\phi\rangle =$ <u>shear</u> χ on
- $\langle \tilde{v}_r \tilde{q}^2\rangle \rightarrow -l^2\epsilon^{1/2}\partial_x\varepsilon$ spreading, entrainment, SOFT

How, Why ? Cont’d

- D_n, χ regulate P.E. exchange between mean, fluctuations ➔ key role in model

- Mixing Length: $l_{mix} = \dfrac{l_0}{\left[1+\dfrac{l_0^2[\partial_x(n-u)]^2}{\epsilon}\right]^{\kappa/2}} = \dfrac{l_0}{\left[1+\left(l_0^2 \,/\, l_{Rh}^2\right)\right]^{\kappa/2}}$

Physics: “Rossby Wave Elasticity’

i.e. $D \sim \dfrac{\langle\tilde{v}^2\rangle}{\Delta\omega} \rightarrow \langle\tilde{v}^2\rangle \dfrac{\Delta\omega}{\omega_r^2+(\Delta\omega)^2} \approx \langle\tilde{v}_r^2\rangle \dfrac{\Delta\omega}{\omega_r^2}$ for $\Delta\omega < \omega_r$

→ <u>waves</u> enhance memory

→ $\omega_r \sim \nabla\langle q\rangle$ → nonlinear Γ_{PV} vs $\langle q\rangle$ → S-curve

- Soft point: κ → suppression exponent

$\kappa = 1$ doesn't always work

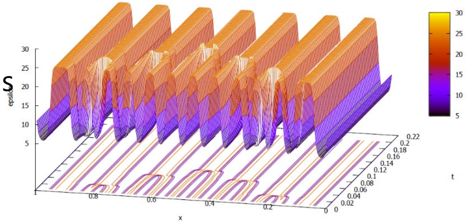
Rigorous bound, from fundamental equations?

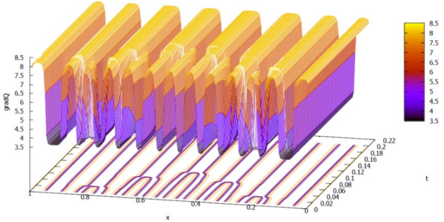
Some Results

Staircase Model – Formation and Merger (QG-HM)

Energy

fluctuations



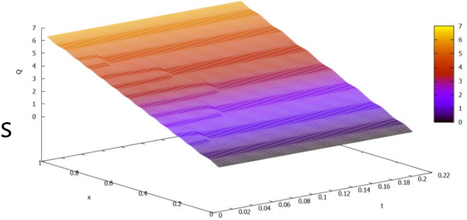


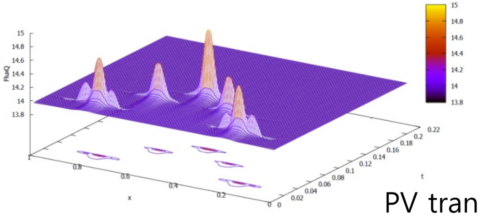
∇q

q

→

mergers





PV transport

- ϵ
- Q_y

} top

- Q
- Γ_q

} bottom

- PV mixing events

Note later staircase mergers induce strong PV flux episodes!

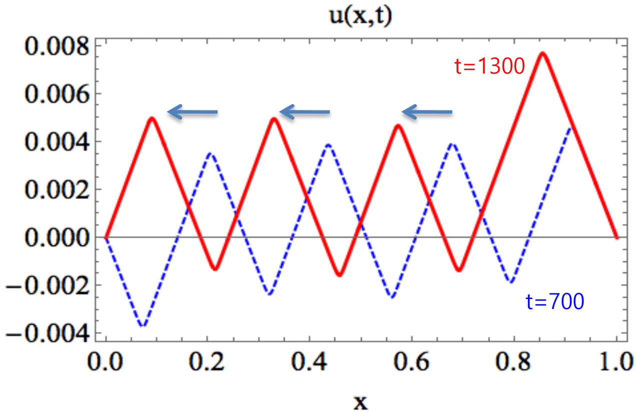
(Malkov, P.D.; PR Fluids 2018)

Staircase are Dynamic Patterns

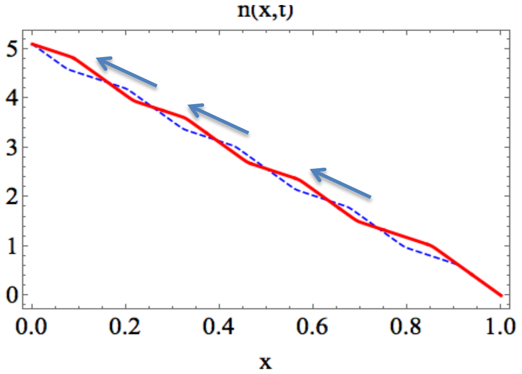
○Shear pattern detaches and delocalizes from its initial position of formation.

○Mesoscale shear lattice moves in the up-gradient direction. Shear layers condense and disappear at x=0.

○Shear lattice propagation takes place over much longer times. From t~O(10) to t~(10^4).



○Barriers in density profile move upward in an "Escalator-like" motion.

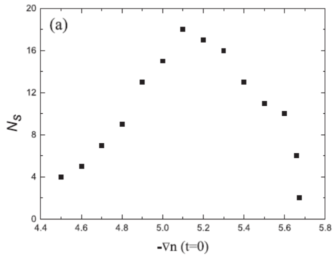


➔ Macroscopic Profile Re-structuring

(Ashourvan, P.D. 2016)

FAQ re: Staircase Structure?

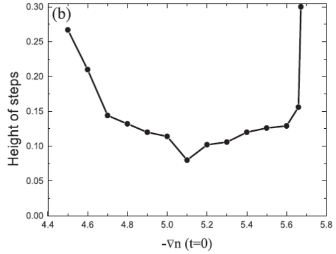
- Number of steps? - domain L → Scale Selection ?!
- Scan # steps vs ∇n at t=0   (n.b. mean gradient)
  - a maximum # steps (and minimal step size) vs ∇n
  - rise: increase in free energy as ∇n ↑
  - drop: diffusive dissipation limits N_s



- Height of steps?
  - minimal height at maximal #

  ➔ system has a ∇n ‘sweet spot’ for many,

  small steps and zonal layers



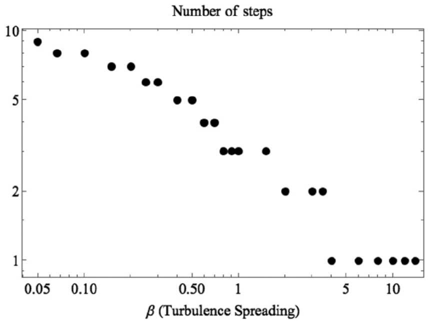
Spreading/Entrainment

- Spreading/entrainment effect on P.E. is unconstrained, beyond $\nabla \cdot \Gamma_q$ structure

Contrast: $D_n, \ \chi$ Follow standard $k - \epsilon$ model CRUDE !

- How robust is staircase to effects of entrainment, avalanching… ?

- $D_\varepsilon \ \rightarrow \ \beta \, l^2 \, \epsilon^{1/2}$

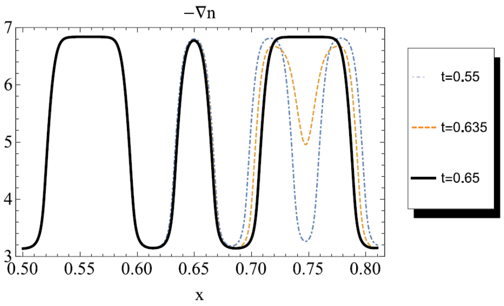


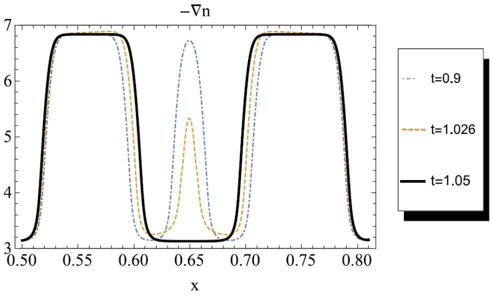
Entrainment has significant effect on S.C. structure

Large β → wash out S.C.

- Important !

Mergers Happen !





- ‘Type-II’ merger (c.f. Balmforth, in ‘Interfaces’)
- ‘Type-I’ (motion) mergers also observed

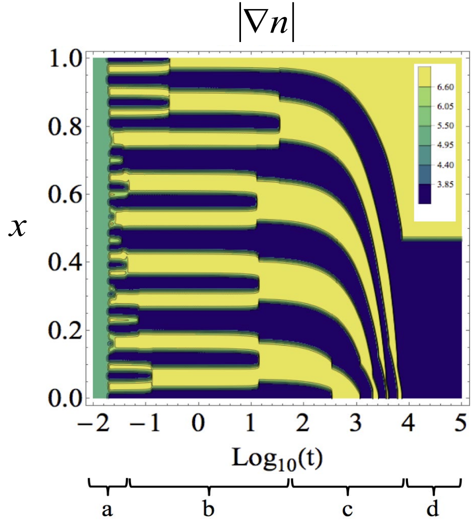
➔ Staircase coarsens….

➔ Obvious TBD:

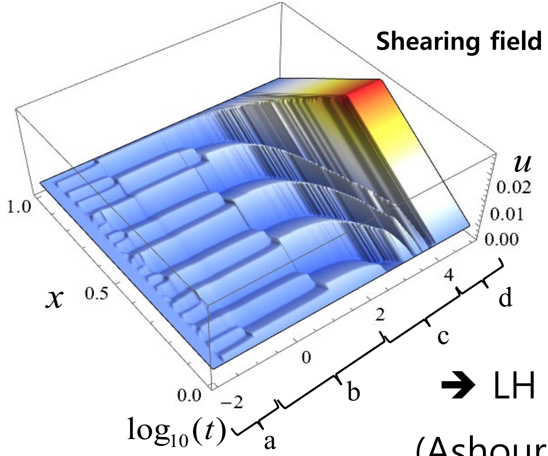
  - Interplay/Competition of Spreading and Mergers?
  - Scan coarsening time vs β, merger rate vs increments in β

Macro-Barriers via Condensation

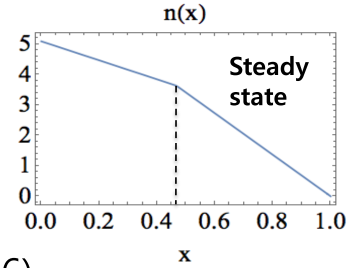
- **(a) Fast merger of micro-scale SC. Formation of meso-SC.**
- **(b) Meso-SC coalesce to barriers**
- **(c) Barriers propagate along gradient, condense at boundaries**
- **(d) Macro-scale stationary profile**



Shearing field



➔ LH transition?



Steady state

(Ashourvan, P.D. 2016)

Conclusion and Current Research

Conclusions

- Shear Flows → externally and self-generated → effective at regulating transport via scale and rate selection
- Potential Vorticity and its conservation are powerful formulations for GFD and Magnetized Plasma dynamics
- Zonal Flows are self-generated flows of minimum inertia, damping and transport and thus are of great interest
- Turbulence, zonal models (and profile corrugations) are a multi-state self-regulating system

Conclusions, cont’d

- Inhomogeneous mixing produces layered domains or staircases → scale selection
- Staircase can be recovered via bi-stable mixing model for $\langle n\rangle, \langle \nabla^2\phi\rangle, \varepsilon$ → emergent length [Rhines scale] is crucial
- Edge Barriers recovered from hierarchical mergers and staircase condensation

Ongoing Research

- Staircase-avalanche co-existence

  ⬇

- Staircase “resilience”
- Heterogeneous staircase → profile, $\langle E_r \rangle$ variation
- Development of coarsening
- Transitions, especially barriers
- Self-organized, ~ marginal cells of pinned turbulence

➔ Rosenbluth ‘87