# Elastic Turbulence in Flatland:
# A Tale of Blobs, Barriers, and Inhomogeneous Mixing

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# Outline

- Elastic Fluids:

  Elasticity ⬄ Memory ⬄ Transport

- Active Scalar Transport in 2D MHD:

  Conventional Wisdom

- New Development: Blobs and Barriers
- Revisiting Quenching
- Inhomogeneous Mixing and Staircases
- Open Questions

# Elastic Fluids

- Interal DOF exerting restoring force on fluid → "springiness"
- Examples:
  - MHD → $\vec{B},\ \vec{J}\times\vec{B}$

    Polymer hydro → Elastic element oldroyd-B
  - Spinodal Decomposition (CHNS) → droplet surface tension
- Elasticity → <u>Memory</u> → Impact on mixing?!

# Active Scalar Transport in 2D MHD: Background and Conventional Wisdom

# Physics: Active Scalar Transport

- Magnetic diffusion, $A$ transport are cases of active scalar transport
- (Focus: 2D MHD) (Cattaneo, Vainshtein ’92, Gruzinov, P. D. ’94, ’95)

scalar mixing – the usual

$$\partial_t A + \nabla\phi \times \hat{z} \cdot \nabla A = \eta \nabla^2 A$$

$$\partial_t \nabla^2 \phi + \nabla\phi \times \hat{z} \cdot \nabla\nabla^2 \phi = \nabla A \times \hat{z} \cdot \nabla\nabla^2 A + \nu \nabla^2 \nabla^2 \phi + \tilde{f}$$

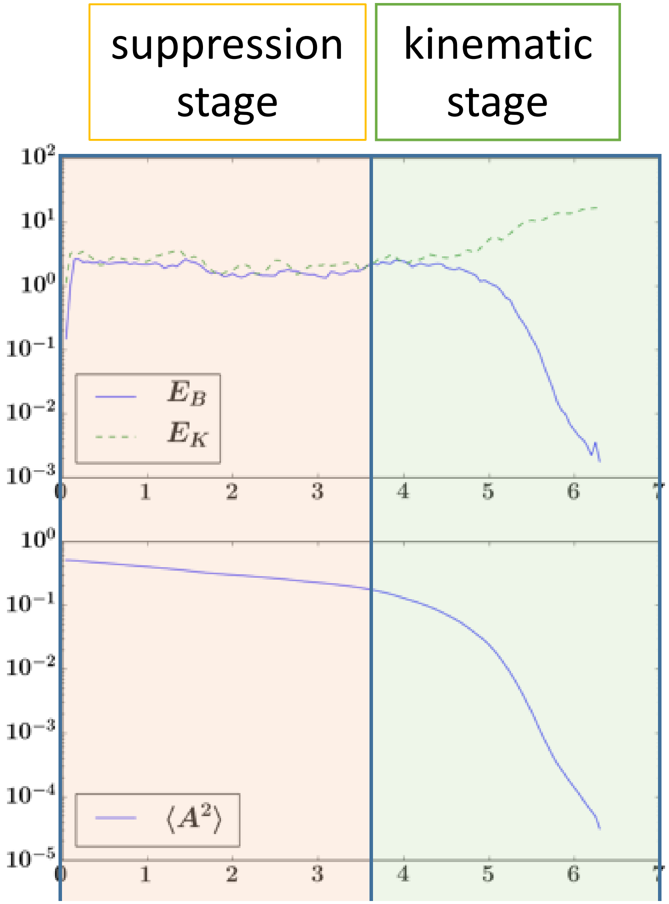
turbulent resistivity

back-reaction

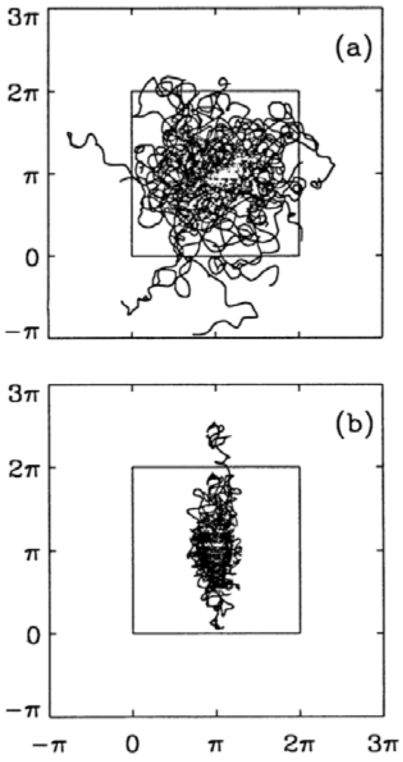
- Seek $\langle v_x A \rangle = -D_T \frac{\partial \langle A \rangle}{\partial x} - \eta \frac{\partial \langle A \rangle}{\partial x}$
- Point: $D_T \neq \sum_{\vec{k}} |v_{\vec{k}}|^2 \, \tau^K_{\vec{k}}$ , often substantially less than kinematic
- Why: <u>Memory</u>! ⟷ Freezing-in
- Cross Phase

# Two Stage Evolution:

- 1. The suppression stage: the (large scale) magnetic field is sufficiently strong so that the diffusion is suppressed.
- 2. The kinematic decay stage: the magnetic field is dissipated so the diffusion rate returns to the kinematic rate.
- Suppression is due to the memory induced by the magnetic field.



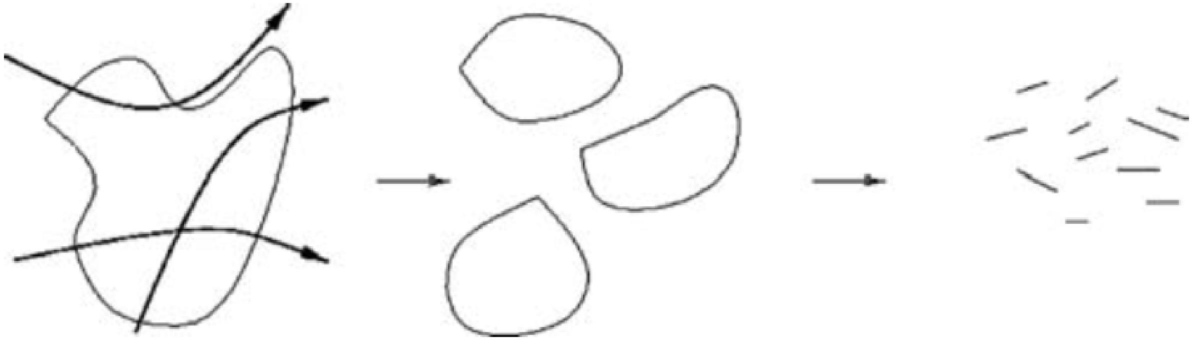
# Conventional Wisdom



- [Cattaneo and Vainshtein 1991]: turbulent transport is suppressed, even for a weak large scale magnetic field is present.
- Starting point: $\partial_t \langle A^2 \rangle = -2\eta \langle B^2 \rangle$
- Assumptions:
  - Energy equipartition: $\frac{1}{\mu_0 \rho} \langle B^2 \rangle \sim \langle v^2 \rangle$
  - Average B can be estimated by: $|\langle \mathbf{B} \rangle| \sim \sqrt{\langle A^2 \rangle}/L_0$
- Define Mach number as: $M^2 = \langle v_A \rangle^2 / \langle \tilde{v}^2 \rangle = \langle v^2 \rangle / v_A^2 = \langle v^2 \rangle / \frac{1}{\mu_0 \rho} \langle B^2 \rangle$
- Result for suppression stage: $\eta_T \sim \eta M^2$
- Fit together with kinematic stage result: $\boxed{\eta_T \sim \frac{ul}{1 + \mathrm{Rm}/M^2}}$
- Physics interpretation of $\eta_T$ ?

# Origin of Memory?

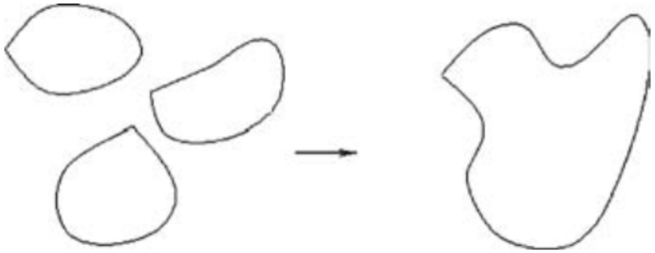
- (a) flux advection vs flux coalescence
  - intrinsic to 2D MHD (and CHNS)
  - rooted in <u>inverse cascade</u> of $\langle A^2 \rangle$ - dual cascades
- (b) tendency of (even weak) *<u>mean</u>* magnetic field to "Alfvenize" turbulence [cf: vortex disruption feedback threshold!]
- Re (a): Basic physics of 2D MHD



Forward transfer: fluid eddies chop up scalar $A$.

# Memory Cont'd

- v.s.



Inverse transfer: current filaments and *A*-blobs attract and coagulate.

- Obvious analogy: straining vs coalescence
- Upshot: closure calculation yields:

$$\Gamma_A = -\sum_{\vec{k}'}\left[\tau_c^{\phi}\langle v^2\rangle_{\vec{k}'} - \tau_c^{A}\langle B^2\rangle_{\vec{k}'}\right]\frac{\partial\langle A\rangle}{\partial x} + \cdots$$

flux of potential

competition

scalar advection vs. coalescence ("negative resistivity")

(+) (-)

N.B.:
Coalescence
→ Negative diffusion
→ Bifurcation

# Conventional Wisdom, Cont'd

- Then calculate $\langle B^2\rangle$ in terms of $\langle v^2\rangle$ (after Zeldovich)

$$\partial_t A + \mathbf{v}\cdot\nabla A = -v_x \frac{\partial\langle A\rangle}{\partial x} + \eta\nabla^2 A$$

- Multiplying by $A$ and sum over modes:

$$\frac{1}{2}[\cancel{\partial_t\langle A^2\rangle} + \cancel{\langle\nabla\cdot(\mathbf{v}A^2)\rangle}] = -\Gamma_A \frac{\partial\langle A\rangle}{\partial x} - \eta\langle B^2\rangle \qquad \frac{\partial\langle A\rangle}{\partial x} \to B_0$$

Dropped stationary case

Dropped periodic boundary → introduce nonlocality?!

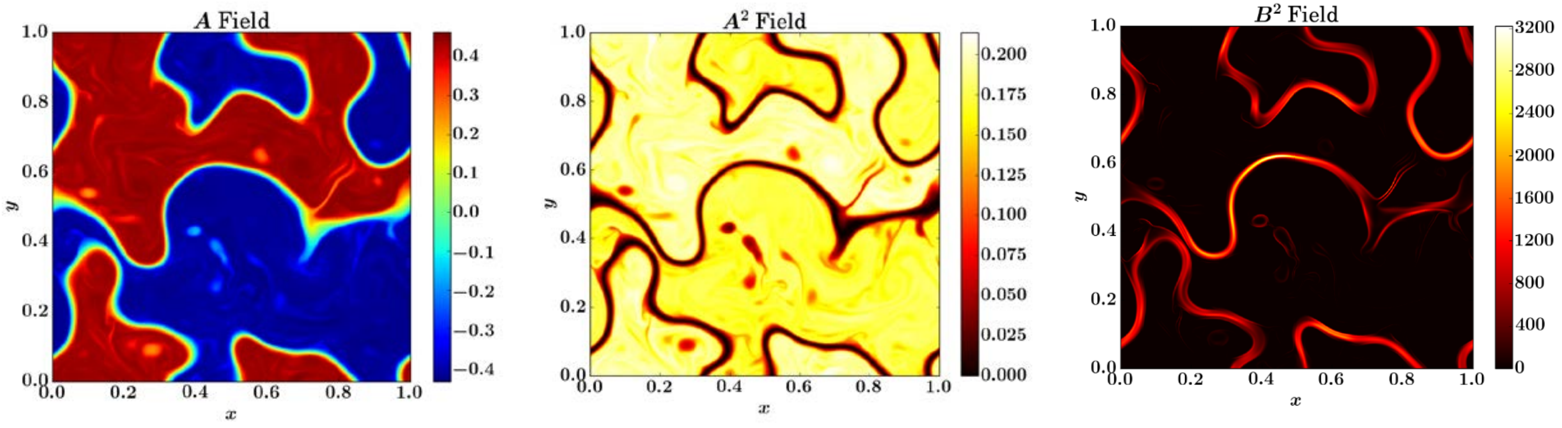
- Therefore: $$\langle B^2\rangle = -\frac{\Gamma_A}{\eta}\frac{\partial\langle A\rangle}{\partial x} = \frac{\eta_T}{\eta}B_0^2$$
- Define Mach number as: $$M^2 \equiv \langle v^2\rangle/v_{A0}^2 = \langle v^2\rangle/(\frac{1}{\mu_0\rho}B_0^2)$$
- Result: $$\boxed{\eta_T = \frac{\sum_{\mathbf{k}} \tau_c\langle v^2\rangle_{\mathbf{k}}}{1+\mathrm{Rm}/M^2} = \frac{ul}{1+\mathrm{Rm}/M^2}}$$
- This theory is not able to describe $B_0 \to 0$ limited!

# New Wrinkles

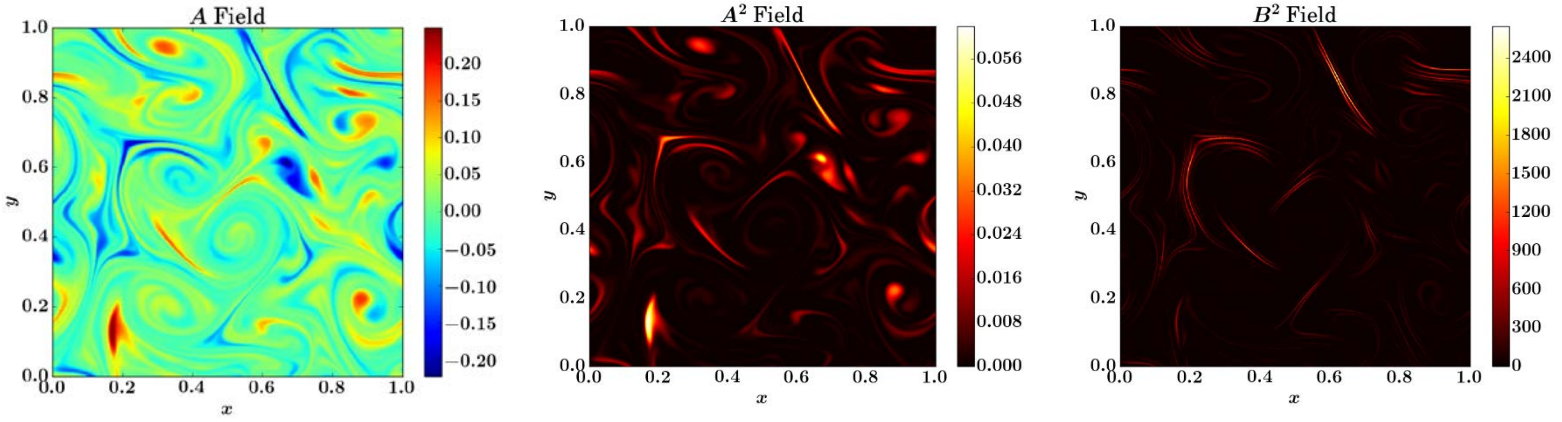
# New Observations

- With no imposed $B_0$, in suppression stage:

Field
Concentrated!



- v.s. same run, in kinematic stage (trivial):



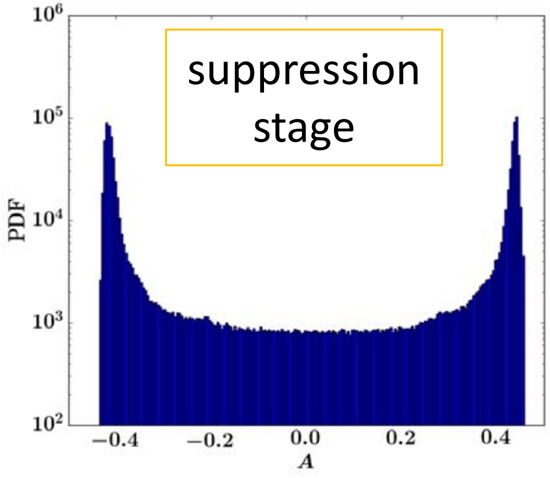
# New Observations Cont'd

- Nontrivial structure formed in real space during the suppression stage.
- $A$ field is evidently composed of "blobs".
- The low $A^2$ regions ←→ 1-dimensional.
- The high $B^2$ regions are strongly correlated with low $A^2$ regions, and also ←→ 1-dimensional.
- 1-dimensional high $B^2$ regions ``barriers''. There, mixing is sharply reduced, relative to $\eta_K$.

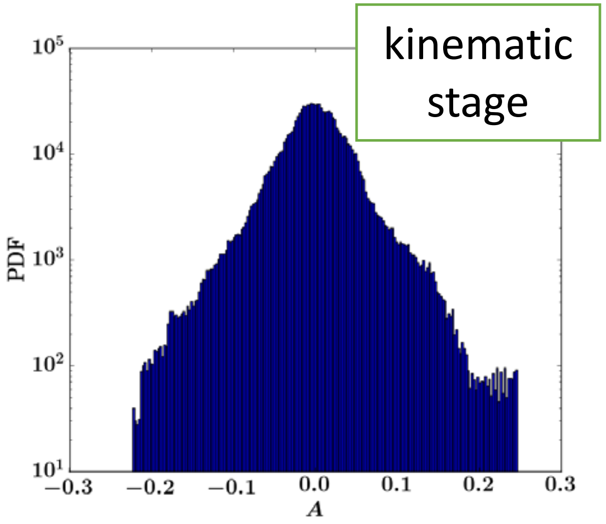
➔ Story one of 'blobs and barriers'

# Evolution of PDF of A

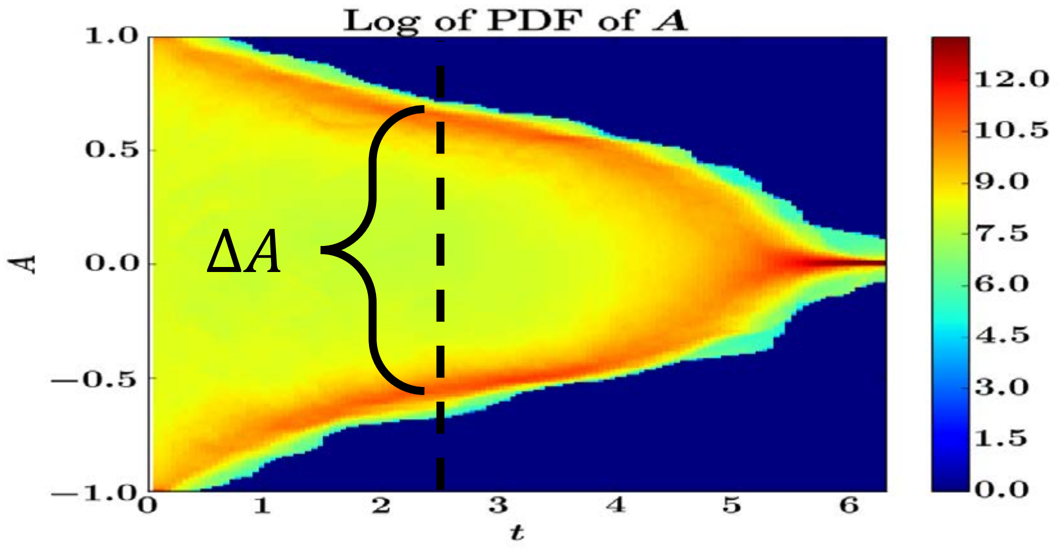
- Probability Density Function (PDF) in two stage:



suppression stage



kinematic stage

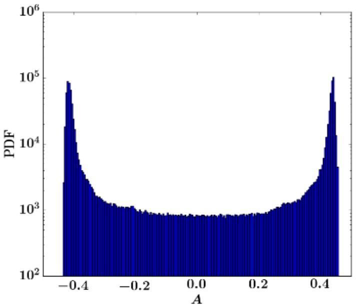


- Time evolution: horizontal "Y".

- The PDF changes from double peak to single peak as the system evolves from the suppression stage to the kinematic stage.

# Unimodal Initial Condition

- One may question whether the bimodal PDF feature is purely due to the initial condition. The answer is No.
- Two non-zero peaks in PDF of A still arise, even if the initial condition is unimodal.



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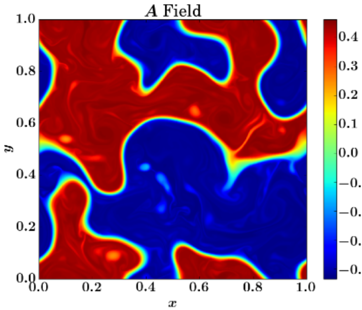

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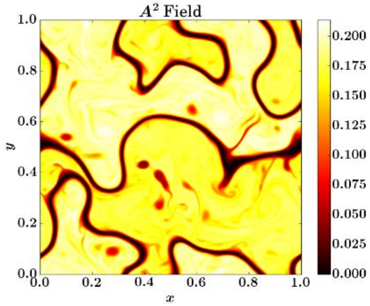

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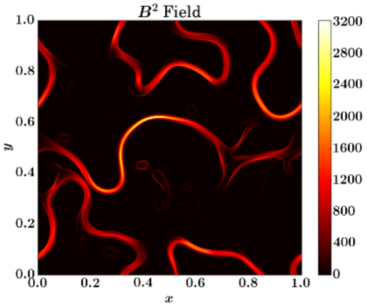
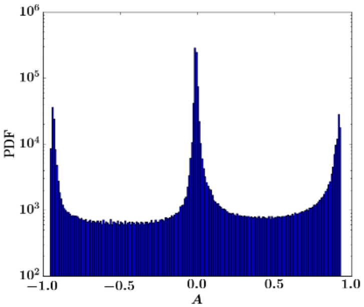


(a4)



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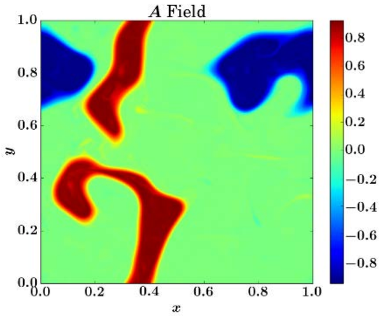

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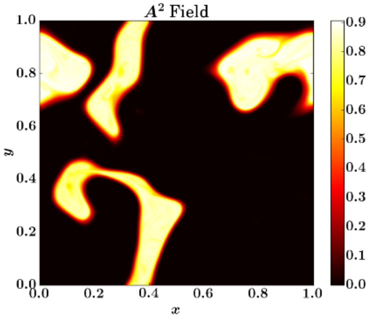

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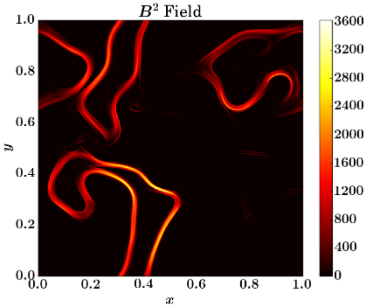
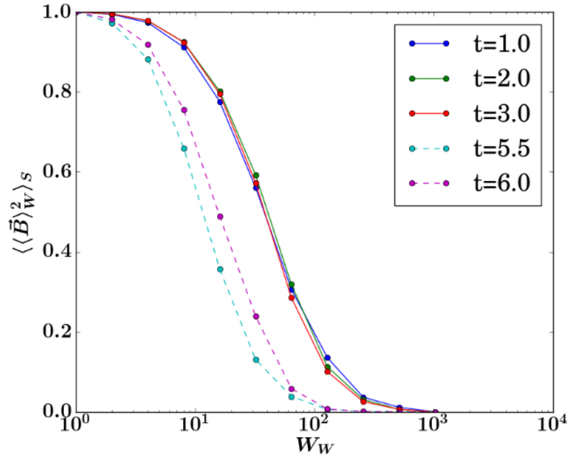
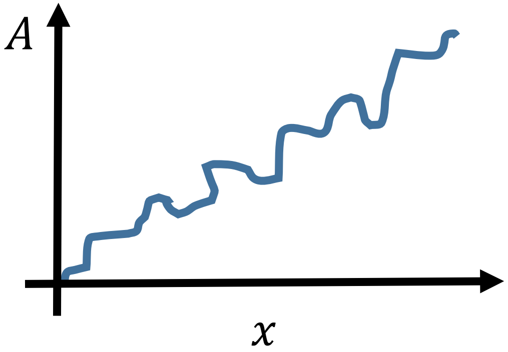


(b4)

# The problem of the mean field $\langle \boldsymbol{B} \rangle$ → What does "Mean" mean?

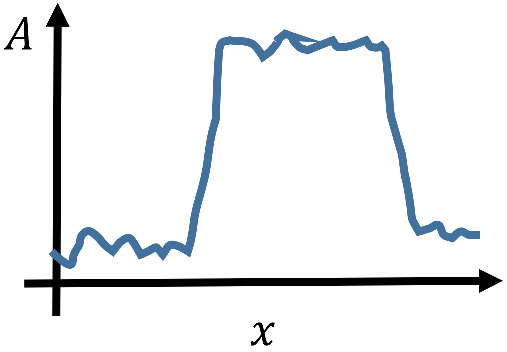
- $\langle B \rangle$ depends on the averaging window.
- With no imposed external field, B is highly intermittent, therefore $\langle B \rangle$ is not well defined.





$|\langle B \rangle| \sim \sqrt{\langle A^2 \rangle}/L_0$ ✓

v.s.



$\langle B \rangle$ not well defined

Reality

# Revisiting Quenching

# New Understanding

- Summary of important length scales: $l < L_{stir} < L_{env} < L_0$
  - System size $L_0$
  - Envelope size $L_{env}$ → emergent (blob)
  - Stirring length scale $L_{stir}$
  - Turbulence length scale $l$, here we use Taylor microscale $\lambda$
  - Barrier width $W$ → emergent
- Quench is not uniform. Transport coefficients differ in different regions.
- In the regions where magnetic fields are strong, $Rm/M^2$ is dominant. They are regions of ***barriers***.
- In other regions, i.e. Inside blobs, $Rm/M'^2$ is what remains. $M'^2 \equiv \langle V^2 \rangle / \left( \frac{1}{\rho} \langle A^2 \rangle / L_{env}^2 \right)$

# New Understanding, cont'd

- From $\partial_t \langle A^2 \rangle = -\langle \mathbf{v} A \rangle \cdot \nabla \langle A \rangle - \nabla \cdot \langle \mathbf{v} A^2 \rangle - \eta \langle B^2 \rangle$
- Retain 2nd term on RHS. Average taken over an envelope/blob scale.
- Define diffusion (closure):

$$\langle \mathbf{v} A \rangle = -\eta_{T1} \nabla \langle A \rangle$$
$$\langle \mathbf{v} A^2 \rangle = -\eta_{T2} \nabla \langle A^2 \rangle$$

- Plugging in: $\partial_t \langle A^2 \rangle = \eta_{T1} (\nabla \langle A \rangle)^2 + \nabla \eta_{T2} \cdot \nabla \langle A^2 \rangle - \eta \langle B^2 \rangle$
- For simplicity: $\langle B^2 \rangle \sim \frac{\eta_T}{\eta} (\langle B \rangle^2 + \langle A^2 \rangle / L_{env}^2)$
- $L_{env}$ is the envelope size. Scale of variation of $\langle A^2 \rangle$.
- Define new strength parameter: $M'^2 \equiv \langle v^2 \rangle / (\frac{1}{\mu_0 \rho} \langle A^2 \rangle / L_{env}^2)$
- Result: $\boxed{\eta_T = \frac{ul}{1 + \mathrm{Rm}/M^2 + \mathrm{Rm}/M'^2} = \frac{ul}{1 + \mathrm{Rm} \frac{1}{\mu_0 \rho} \langle \mathbf{B} \rangle^2 / \langle v^2 \rangle + \mathrm{Rm} \frac{1}{\mu_0 \rho} \langle A^2 \rangle / L_{env}^2 \langle v^2 \rangle}}$

$$\eta_T = V\,l\,/\left[1+\frac{R_m}{M^2}+\frac{R_m}{M'^2}\right]$$

- Barriers:

Strong field

$$\eta_T \approx V\,l\,/\left[1+R_m\frac{\langle B\rangle^2}{\rho\langle\tilde{V}^2\rangle}\right]$$

- Blobs:

Weak effective field

$$\eta_T \approx V\,l\,/\left[1+R_m\frac{\langle A^2\rangle}{\rho L_{env}^2\,\langle\tilde{V}^2\rangle}\right]$$

- Quench stronger in barriers, highly non-uniform

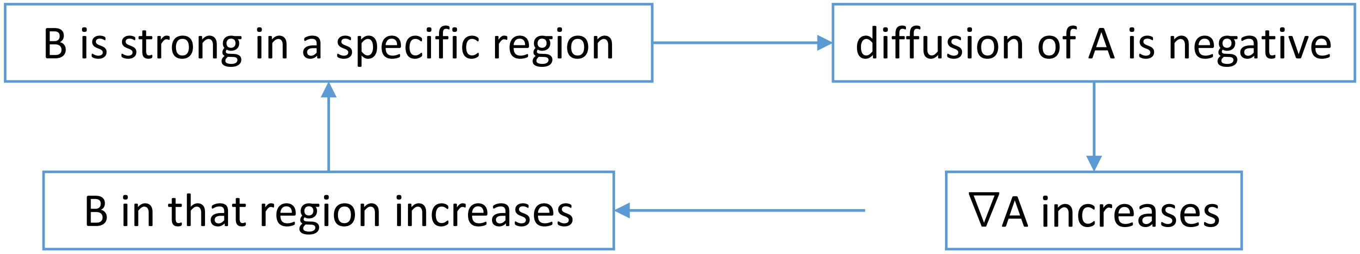
# Formation of Barriers

- How do the barriers form?

flux coalescence

$$\eta_T = \sum_{\mathbf{k}} \tau_c[\langle v^2\rangle_{\mathbf{k}} - \frac{1}{\mu_0\rho}\langle B^2\rangle_{\mathbf{k}}]$$

- From above, strong B regions can support negative incremental $\eta_T = \delta\Gamma_A/\delta(-\nabla A) < 0$, suggesting clustering
- $\langle\eta_T\rangle > 0$
- Positive feedback: a twist on a familiar theme



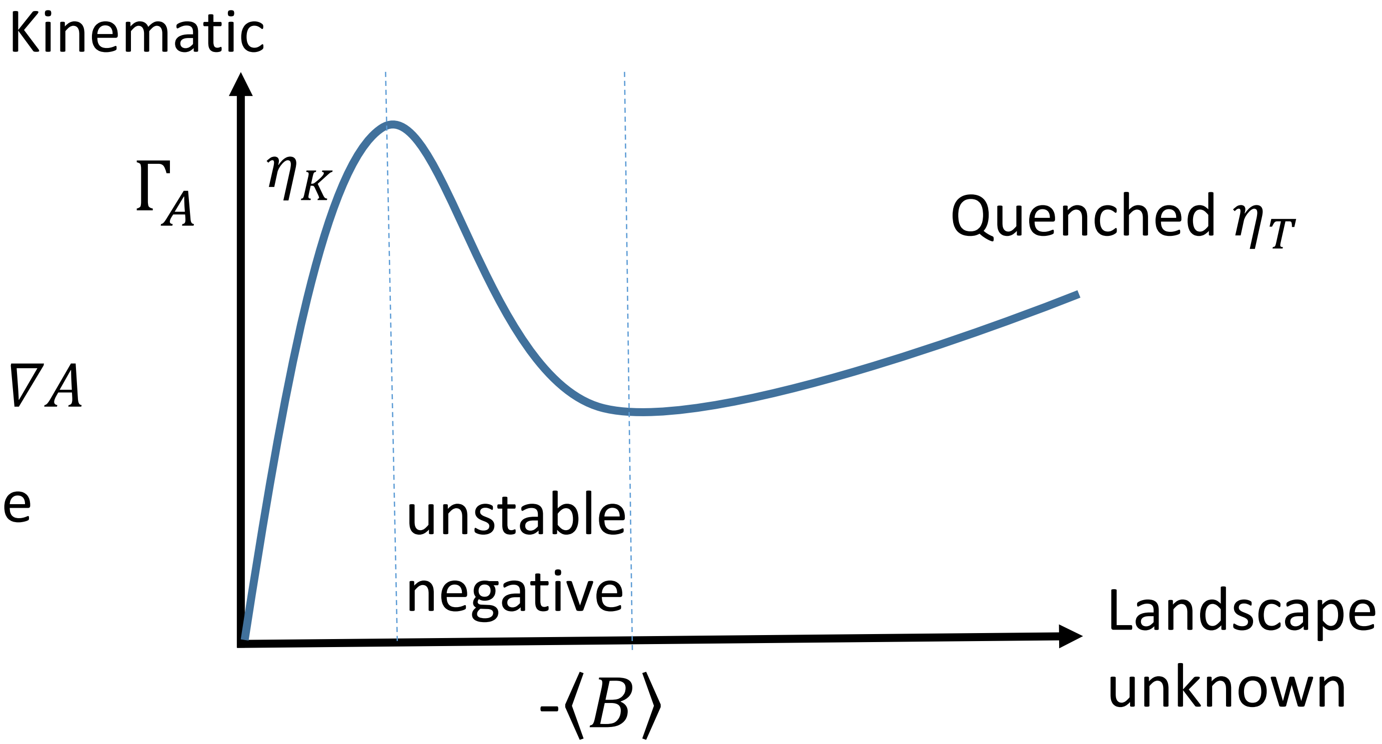
# Formation of Barriers, Cont’d

- Negative resistivity leads to barrier formation.
- The S-curve due to the nonlinear dependence of $\Gamma_A$ on B.
- When slope negative → negative (incremental) resistivity.

Barriers ←→

Bistability of $\Gamma_A$ vs $\nabla A$

→ a familiar theme



# Describing the Barriers

- How to measure the barrier width $W$.
- Starting point: $W \sim \Delta A / B_b$
- Use $\sqrt{\langle A^2 \rangle}$ to calculate $\Delta A$
- Define the barrier regions as: $B(x,y) > \sqrt{\langle B^2 \rangle} * 2$

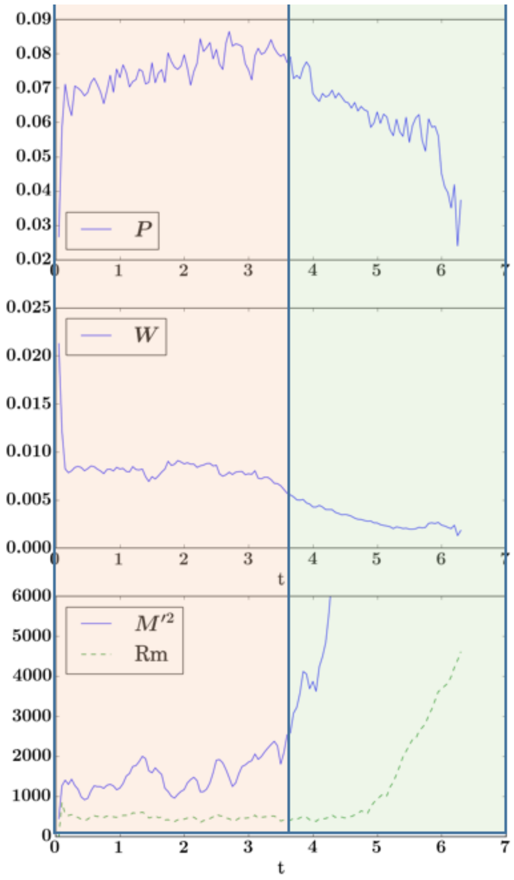
arbitrary threshold

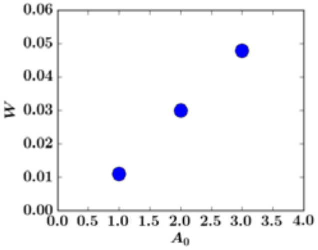
- Define barrier packing fraction: $P \equiv \frac{\#\text{ of grid points for barrier regions}}{\#\text{ of total grid points}}$
- Use use the magnetic fields in the barrier regions to calculate the magnetic energy: $\boxed{\sum_{\text{barriers}} B_b^2 \sim \sum_{\text{system}} B^2}$
- Thus $\langle B_b^2 \rangle \sim \langle B^2 \rangle / P$
- So barrier width can be estimated by: $\boxed{W^2 \equiv \langle A^2 \rangle / (\langle B^2 \rangle / P)}$

N.B. All magnetic energy in the barriers

# Describing the Barriers

- Time evolution of $P$ and $W$:
  - P, W collapse in decay
  - $M'$ rises
- Sensitivity of $W$:
  - $A_0$ or $1/\mu_0\rho$ greater → $W$ greater;
  - $f_0$ greater, $W$ smaller; (ala' Hinze)
  - $W$ not sensitive to $\eta$ or $\nu$.





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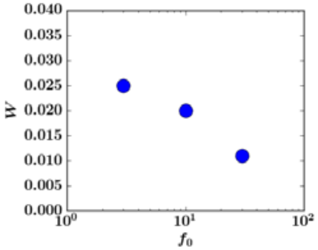

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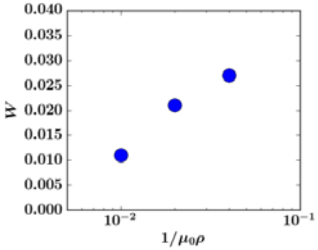

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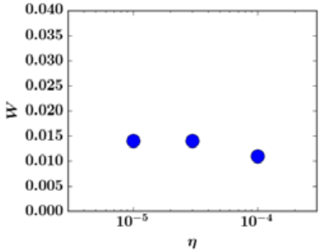

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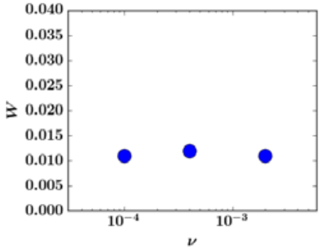
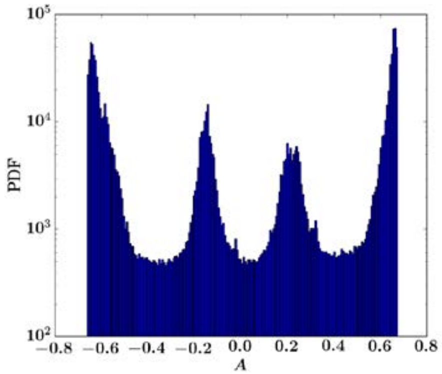


(e)

# Active Scalar Staircases

# Staircase (inhomogeneous Mixing, Bistability)

- Staircases emerge spontaneously! – <u>Barrier lattices</u>
- Initial condition is the usual cos function (bimodal)
- The only major sensitive parameter (from runs above) is the forcing scale k=32 (for all runs above k=5).
- Resembles the PV staircase



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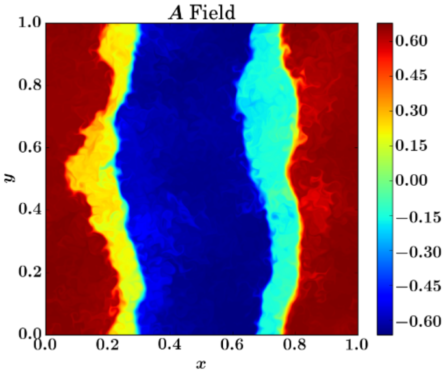

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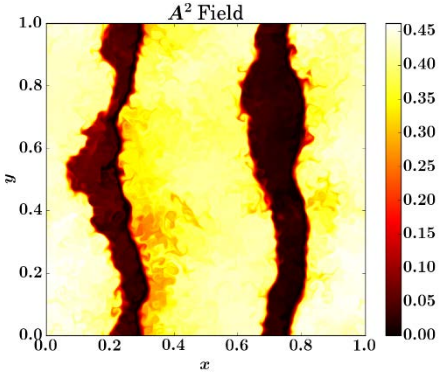

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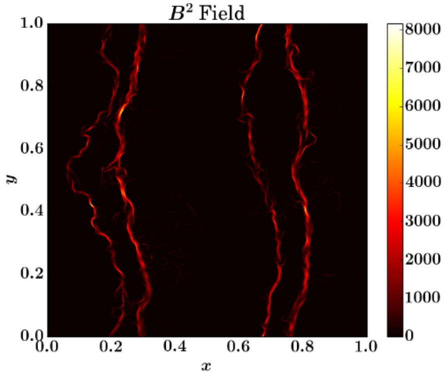

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# Conclusions / Summary

- Magnetic fields suppress turbulent diffusion in 2D MHD by: formation of intermittent ***<u>transport barriers</u>***.
- Magnetic structures:
  - Barriers – thin, 1D strong field regions
  - Blobs – 2D, weak field regions
- Quench not uniform:

$$\eta_T = \frac{ul}{1+\mathrm{Rm}\frac{1}{\mu_0\rho}\langle\mathbf{B}\rangle^2/\langle v^2\rangle+\mathrm{Rm}\frac{1}{\mu_0\rho}\langle A^2\rangle/L_{env}^2\langle v^2\rangle}$$

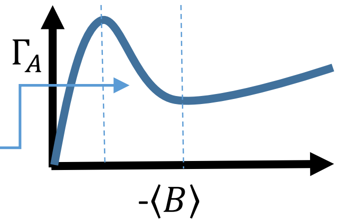
barriers, strong B

blobs, weak B, $\nabla^2\langle A^2\rangle$ remains

- Barriers form due to negative resistivity:

$$\eta_T = \sum_{\mathbf{k}}\tau_c[\langle v^2\rangle_{\mathbf{k}} - \frac{1}{\mu_0\rho}\langle B^2\rangle_{\mathbf{k}}]$$

flux coalescence



- Formation of “magnetic staircases” observed for some stirring scale

# General Conclusions (MHD and CHNS)

- Dual (or multiple) cascades can interact with each other, and can modify one another. N.B.: Focus on $\langle A^2 \rangle$ transfer.
- We show how a length scale, e.g. the Hinze scale in 2D CHNS, emerges from the balance of kinetic energy and elastic energy in blobby turbulence. → blob scale in MHD?!
- Negative incremental resistivity can exist in a simple system such as 2D MHD. This results in the formation of nontrivial real space structure.

# Future Works

- Extension of the transport study in MHD:
  - Numerical tests of the new $\eta_T$ expression ?
  - What determines the barrier width and packing fraction ?
  - Why does layering appear when the forcing scale is small ?
  - What determines the step width, in the case of layering
- Other similar systems can also be studied in this spirit. e.g. Oldroyd-B model for polymer solutions. (drag reduction)
- Reduced Model of Magnetic Staircase

See Also:

C. Chen, P.D., Tobias: "PV Mixing in a Tangled Magnetic Field", in preparation

# Reading

Fan, P.D., Chacon:

- PRE Rap Comm 99, 041201 (2019)
- PoP 25, 055702 (2018)
- PRE Rap Comm 96, 041101 (2017)
- Phys Rev Fluids 1, 054403 (2016)

## Thank you!

# Back-Up

# 2D CHNS and 2D MHD

| | 2D MHD | 2D CHNS |
|---|---|---|
| Magnetic Potential | $A$ | $\psi$ |
| Magnetic Field | $\mathbf{B}$ | $\mathbf{B}_\psi$ |
| Current | $j$ | $j_\psi$ |
| Diffusivity | $\eta$ | $D$ |
| Interaction strength | $\frac{1}{\mu_0}$ | $\xi^2$ |

- 2D CHNS Equations:

$$\partial_t \psi + \vec{v} \cdot \nabla \psi = D \nabla^2 (-\psi + \psi^3 - \xi^2 \nabla^2 \psi)$$

$$\partial_t \omega + \vec{v} \cdot \nabla \omega = \frac{\xi^2}{\rho} \vec{B}_\psi \cdot \nabla \nabla^2 \psi + \nu \nabla^2 \omega$$

$-\psi$: Negative diffusion term

$\psi^3$: Self nonlinear term

$-\xi^2 \nabla^2 \psi$ : Hyper-diffusion term

With $\vec{v}=\hat{z} \times \nabla \phi$, $\omega = \nabla^2 \phi$, $\vec{B}_\psi = \hat{z} \times \nabla \psi$, $j_\psi = \xi^2 \nabla^2 \psi$. $\psi \in [-1,1]$.

- 2D MHD Equations:

$$\partial_t A + \vec{v} \cdot \nabla A = \eta \nabla^2 A$$

$$\partial_t \omega + \vec{v} \cdot \nabla \omega = \frac{1}{\mu_0 \rho} \vec{B} \cdot \nabla \nabla^2 A + \nu \nabla^2 \omega$$

$A$: Simple diffusion term

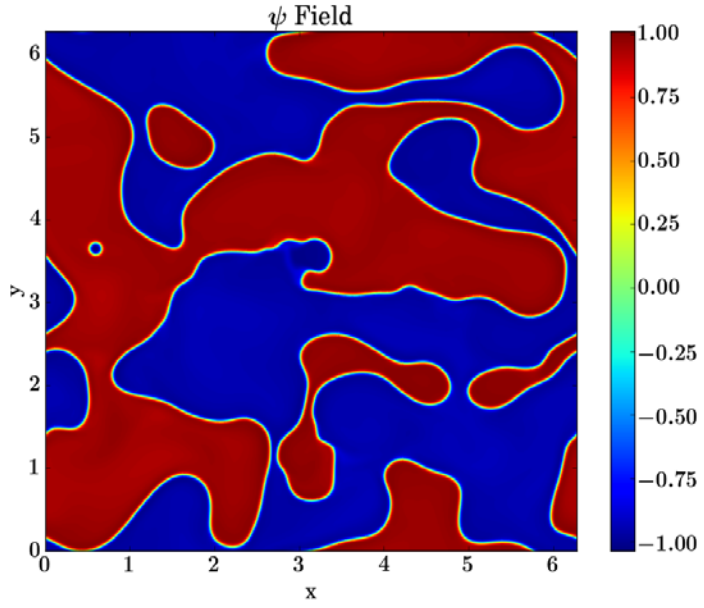
With $\vec{v}=\hat{z} \times \nabla \phi$, $\omega = \nabla^2 \phi$, $\vec{B} = \hat{z} \times \nabla A$, $j = \frac{1}{\mu_0} \nabla^2 A$

See [Fan et.al. 2016] for more about CHNS.

# 2D CHNS and 2D MHD

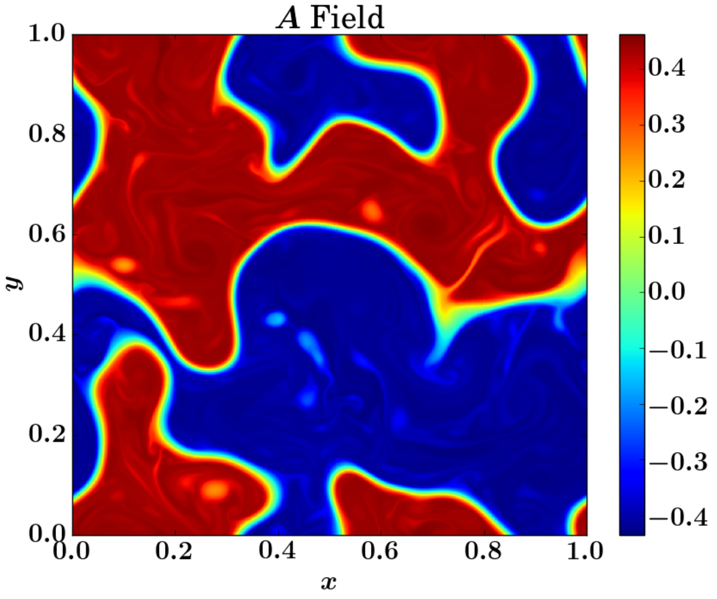
- The $A$ field in 2D MHD in suppression stage is strikingly similar to the $\psi$ field in 2D CHNS (Cahn-Hilliard Navier-Stokes) system:

$\psi$ field in 2D CHNS



v.s.

$A$ field in 2D MHD



# Constitutive Relations → Deborah Number

- J. C. Maxwell:

$$(\text{stress}) + \overset{\substack{\text{relaxation}\\ \downarrow}}{\tau_R}\, \frac{d(\text{stress})}{dt} = \overset{\substack{\text{viscosity}\\ \downarrow}}{\eta}\, \frac{d}{dt}\,(\text{strain})$$

- If $\tau_R/T = D \ll 1$, stress = $\eta \frac{d}{dt}$ (strain)

$T \equiv$ dynamic time scale

$$\Pi = -\,\eta \nabla \vec{v}$$ <u>viscous</u>

- If $\tau_R/T = D \gg 1$, stress $\cong \frac{\eta}{\tau_R}$ (strain)

~ E (strain) <u>elastic</u>

- Limit of “freezing-in”: D≫1 is criterion.

- $D$ ~ Deborah Number ~ $|\nabla V|/\omega_Z$ ~ $\tau_{relax}/\tau_{dyn}$
- Limit for elasticity: $D \gg 1$ → limit for elasticity
- Why “Deborah”? →

Hebrew Prophetess Deborah:

“The moutains flowed before the Lord.” (Judges)

$\therefore$

- Revisit Heraclitus (1500 years later):

“All things flow” – if you can wait long enough

# Simulation Setup

- PIXIE2D: a DNS code solving 2D MHD equations in real space:

$$\partial_t A + \mathbf{v} \cdot \nabla A = \eta \nabla^2 A$$

$$\partial_t \omega + \mathbf{v} \cdot \nabla \omega = \frac{1}{\mu_0 \rho} \mathbf{B} \cdot \nabla \nabla^2 A + \nu \nabla^2 \omega + f$$

- 1024^2 resolution.
- External forcing $f$ is isotropic homogeneous.
- Periodic boundary conditions (both).
- Initial conditions:
  - (1) bimodal: $A_I(x, y) = A_0 \cos 2\pi x$
  - (2) unimodal: $A_I(x, y) = A_0 * \begin{cases} -(x - 0.25)^3 & 0 <= x < 1/2 \\ (x - 0.75)^3 & 1/2 <= x < 1 \end{cases}$