Layering or Homogenization

- A Matter of Time ?!

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Or

"Key Physics of Staircase Formation via Simple Examples"

→ N.B. All herein assembled from single eddy and static superposition there of

→ c.f. X. Fan, P.D., Chacon, Phys Rev E (Rap. Comm) 2017

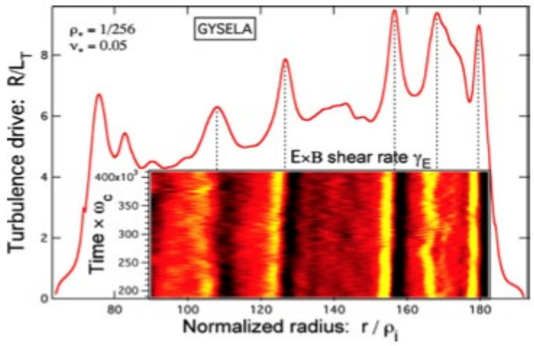
F. Ramirez, P.D., ongoing

Staircase and Layering

- Old story in fluid dynamics, c.f. 1972 O.M. Phillips "Turbulence in a strongly stratified fluid – is it unstable?"

→ transport bifurcation from modulation

- Popular in M.F.E. – especially for color picture from 3D Gyrokinetic Simulations (see this meeting)



Suggested ideas:

- Zonal flow eigenmode
- ExB shear feedback, predator-prey
- Jams (Kosuga, PD, Gurcan)

...

Staircase and Layering, cont'd

- But... basic, minimal physics questions unanswered
- Clue: Staircase formation, dynamics captured in ultra-simple mixing model with 2 scales

Balmforth et. al. '98 (BLY)	Ashourvan, P.D., 2016 (AD)
Kinetic Energy	Guo et. al. 2018
Density	Potential Enstrophy
Ozmidov Scale	Density
	Flow Shear
	Rhines Scale

- N.B. Non-trivial element is emergent scale Ozmidov, Rhines

Some Questions

- How does staircase beat homogenization?

  (i.e. Prandtl – Batchelor Theorem)

- Does the staircase beat homogenization?

  → Or does mixing win, eventually ... ?

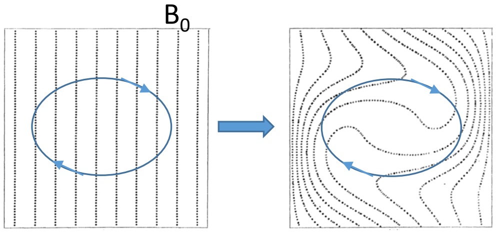
- Is the staircase a meta-stable state?
- What is the minimal set of scales to recover layering?

Part I:

What of a Single Eddy?

Flux Expulsion

- Simplest dynamical problem in MHD (Weiss '66, et. seq.)
- Closely related to "PV Homogenization" - 2D Fluid

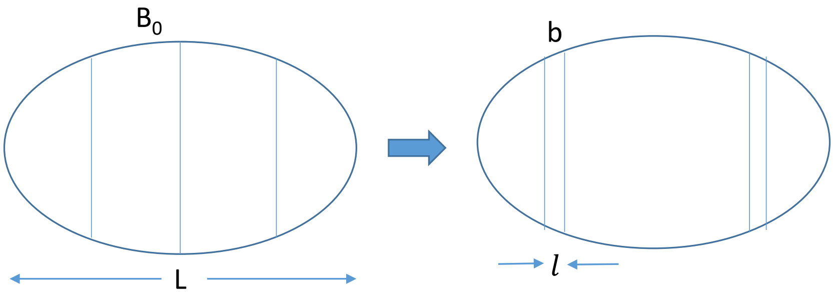


Rm~vL/$\eta \gg 1$

n.b. sheared flow

- Field wound-up, "expelled" from eddy
- For large Rm, field concentrated in boundary layer of eddy
- Ultimately, back-reaction asserts itself for sufficient B_0
- End state: Mixing

Simple Description:



after n turns:
nl=L

- Flux conservation: $B_0L \sim bl$ Wind up: $b = nB_0$ (field stretched)
- Rate balance: wind-up ~ dissipation

$$\frac{v}{L} B_0 \sim \frac{\eta}{l^2} b \; . \quad \tau_{expulsion} \sim \left(\frac{L}{v_0}\right) Rm^{1/3} .$$

$$l \sim \delta_{BL} \sim L/Rm^{1/3} \; . \quad b \sim Rm^{1/3} B_0 \; .$$

N.B. differs from Sweet-Parker!

→ Time scales:

L/V_0 - eddy rotation

$(L/V_0)R_m^{1/3}$ - expulsion

L^2/η - diffusion

What's the Physics? – Shearing + Dissipation!

- Shear dispersion! (Moffatt, Kamkar '82, Dupree '66)

  $\partial_t A + \vec{v} \cdot \nabla A = \eta \nabla^2 A$

  then $v_y = v_y(x) = v_{y0} + x v_y' + \cdots$ (shearing coordinates)

  $\frac{dk_x}{dt} = -k_y v_y'$, $\frac{dk_y}{dt} = 0$ (k tilts)

- $\partial_t A + x v_y' \partial_y A - \eta \left(\partial_x^2 + \partial_y^2 \right) A = 0$

  $A = A(t) \exp i(\vec{k}(t) \cdot \vec{x})$; $A = A(0) \exp \left[-\frac{k_y^2 V_y'^2 \eta t^3}{3} \right]$

- $\tau_{mix} \cong \tau_{shear} Rm^{1/3} = (v_y'^{-1}) Rm^{1/3}$

N.B.: Shear promotes mixing!

Prove it ! (Prandtl, Batchelor)

$\partial_t A + \vec{v} \cdot \nabla A = \nabla \cdot (\eta \nabla A)$

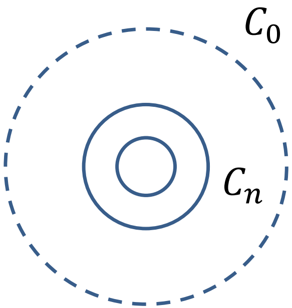
$t \to \infty$

$\int_{A_n} \vec{v} \cdot \nabla A d^2x = \int_{A_n} d^2x \nabla \cdot \eta \cdot \nabla A$

LHS = $\int_{A_n} d^2x \, \nabla \cdot [\vec{v} A] = \int dl \, \hat{n}_{C_n} \cdot (\vec{v} A) = 0$

$\hat{n} \cdot \vec{v} = 0$, by definition

RHS = $\int_{C_n} dl \, \eta \, \hat{n} \cdot \nabla A = \int_{C_n} dl \, \eta \, \hat{n} \cdot \nabla \phi_n \, \delta A / \delta \phi_n$, any n



$A \to$ const within C_0 as $t \to \infty$, nested streamlines

Prove it, cont'd

So RHS = $\eta \frac{\delta A}{\delta \phi_n} \underbrace{\int dl\, \hat{n} \cdot \nabla \phi_n}_{\Gamma_n}$

RHS = 0, for all n ➔ $\frac{\delta A}{\delta \phi} = 0$

→ So A mixed, homogenized

$\nabla A \rightarrow \vec{B}$ expelled to boundary

→ N.B.:

- Critical elements: conservative dissipation

  nested closed streamlines

- Proof not reveal time scale !

A Very Small Upgrade:

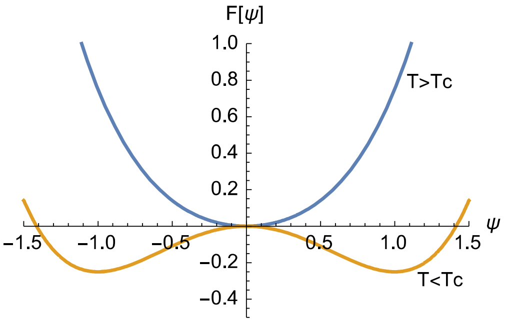
Passive Cahn-Hilliard Advection

A Brief Derivation of the CHNS Model

- Second order phase transition → Landau Theory.
- Order parameter: $\psi(\vec{r},t) \stackrel{\text{def}}{=} [\rho_A(\vec{r},t) - \rho_B(\vec{r},t)]/\rho$
- Free energy:

binary mixture

$$F(\psi) = \int d\vec{r}\left(\underbrace{\frac{1}{2}C_1\psi^2 + \frac{1}{4}C_2\psi^4}_{\text{Phase Transition}} + \underbrace{\frac{\xi^2}{2}|\nabla\psi|^2}_{\text{Gradient Penalty}}\right)$$



- $C_1(T), C_2(T)$.
- Isothermal $T < T_C$. Set $C_2 = -C_1 = 1$:

$$F(\psi) = \int d\vec{r}\left(-\frac{1}{2}\psi^2 + \frac{1}{4}\psi^4 + \frac{\xi^2}{2}|\nabla\psi|^2\right)$$

A Brief Derivation of the CHNS Model

- Continuity equation: $\frac{d\psi}{dt} + \nabla \cdot \vec{J} = 0$. Fick's Law: $\vec{J} = -D\nabla\mu$.
- Chemical potential: $\mu = \frac{\delta F(\psi)}{\delta\psi} = -\psi + \psi^3 - \xi^2\nabla^2\psi$. <u>Conservative</u>
- Combining above → Cahn Hilliard equation:

$$\frac{d\psi}{dt} = D\nabla^2\mu = D\nabla^2(-\psi + \psi^3 - \xi^2\nabla^2\psi)$$

- $d_t = \partial_t + \vec{v} \cdot \nabla$. Surface tension: force in Navier-Stokes equation:

$$\partial_t\vec{v} + \vec{v} \cdot \nabla\vec{v} = -\frac{\nabla p}{\rho} - \psi\nabla\mu + \nu\nabla^2\vec{v}$$

- For incompressible fluid, $\nabla \cdot \vec{v} = 0$.

Passive C-H Advection

- $\partial_t \psi + \vec{v} \cdot \nabla \psi = \nabla \cdot D \nabla \delta F(\psi)/\delta\psi = D \nabla^2 (-\psi + \psi^3 - \varepsilon^2 \nabla^2 \psi)$

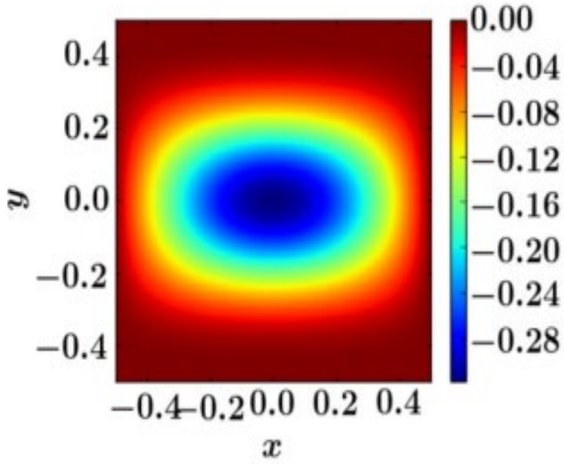
single eddy, sheared flow, prescribed, ala' expulsion

'anti-diffusion' phase separation

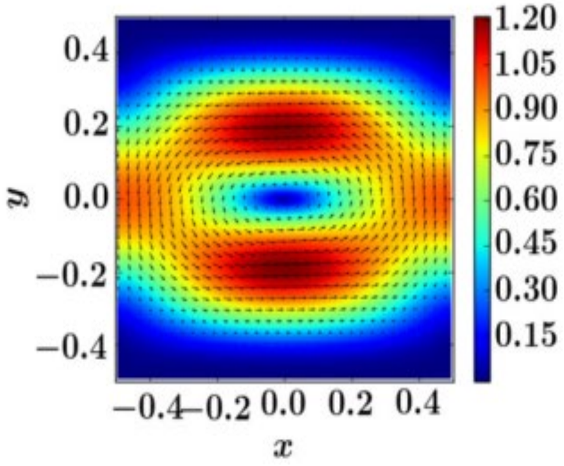
hyper-diffusion (regularization)

- $P_e = VL/D \gg 1\,, \quad C_h = \varepsilon/L \ll 1$
- two free energy minima + anti-diffusion
- no feedback of ψ on flow, No 'turbulence'
- minimal upgrade of scalar problem...

Set Up



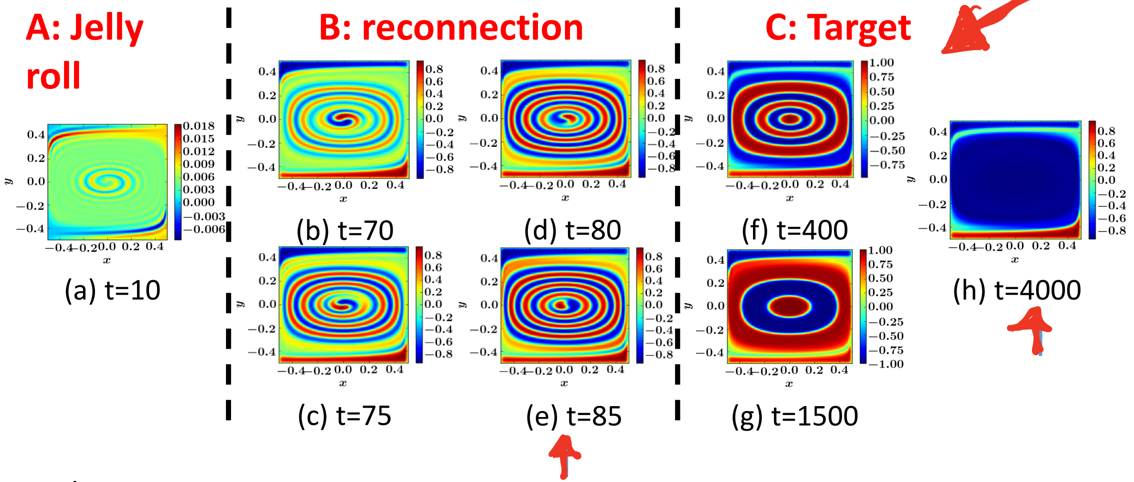
Stream Function



Velocity Field

Single Eddy Mixing -- Cahn-Hilliard

- 3 stages: (A) the *"jelly roll"* stage, (B) the *topological evolution* stage, and (C) the *target pattern* stage.
- ψ ultimately homogenized in slow time scale, but metastable target patterns formed and merge.



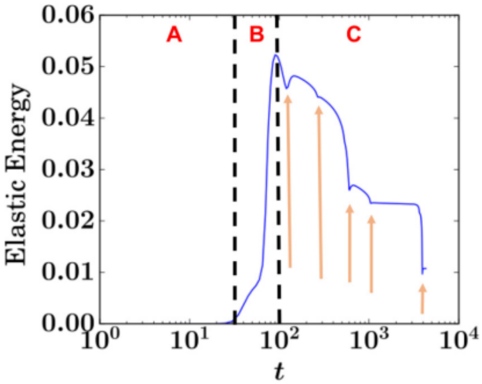
Note coarsening!

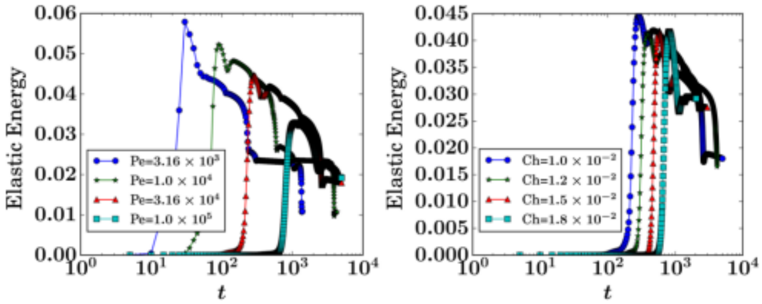
- Additional mixing time emerges.

$\tau_{mix} \sim \tau_0 \,/Pe^{-1/5} C_h^{2/5}$ $\leftrightarrow$ shear + hyper-diffusion

→ clear distinction between reconnection and target phase

Single Eddy Mixing, cont'd

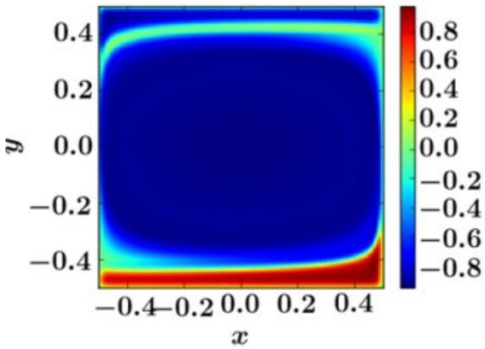




- Layered state persists exponentially long time
- Layer mergers episodic, related to dips in elastic energy
- Long tie mergers occur in axisymmetric, azimuthally homogenized state

➔ Staircase here is <u>meta-stable state</u>

End State – Full Mixing



- end state homogenized
- exponentially long time scale

- Dirichlet B.C.
- → Meta-Stable staircase ultimately <u>decays</u>

Prove it II

- From before:

$$0 = \int_{C_n} dl\, D\, \hat{n} \cdot \nabla[-\psi + \psi^3 - \varepsilon^2 \nabla^2 \psi]$$

$$F = \varepsilon^2 \nabla^2 \psi$$

$$= \int_{C_n} D\hat{n} \cdot \nabla\psi\, [-1 + 3\psi^2] - \int dl\, D\hat{n} \cdot \nabla F$$

$$= \int_{C_n} dl\, D\hat{n} \cdot \nabla\phi_n \left\{[-1 + 3\psi^2] \frac{\delta\psi}{\delta\phi_n} - \frac{\delta F}{\delta\phi_n}\right\}$$

as $F = F(\psi)$

$$= \int_{C_n} dl\, D\, \hat{n} \cdot \nabla\phi_n \left[-1 + 3\psi^2 - \frac{\delta F}{\delta\psi}\right] \delta\psi/\delta\phi_n$$

So $\frac{\delta\psi}{\delta\phi_n} = 0$, all n → homogenization again. Time scales?

Partial Summary

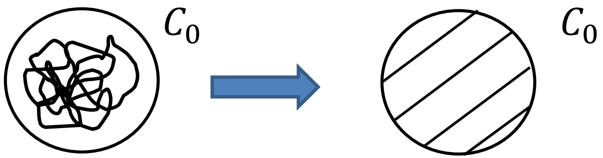
- Passive CH Flow
  - Conservative, anti-diffusion, 2 free energy minima
  - ➔ Simple system
- Generates axisymmetric layered target
- 2 time scales for ~ homogenization
- Layered state meta-stable, long time scale coarsening
- Ultimately homogenized...

Part II:

Cellular Array

Homogenized Vortex as Transport Barrier !?

- PV Homogenization



C_0 - bounded closed streamline

$\tau \sim \tau_{rot} Re^{1/3}$

- Boundary
  - Strong PV gradient
  - Layer width ~ $L \,/\, Re^{1/3}$
  - Strong memory (“Rossby wave elasticity) $\omega \approx k_x \nabla q / k^2$

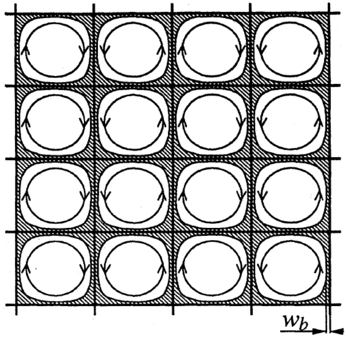
2 scales: $L,\ L/Re^{1/3}$

- Eddy boundary is <u>transport barrier</u>

➔ Layering in array of cells ?!

Consider Cellular Lattice

- Marginally overlapping cells (G.I. Taylor, Moffat, Rosenbluth, Shraiman, ...)
  - Stationary cell pattern
  - Two time rates: $V_0/l_0,\ D_0/l_0^2$
  - $P_e = V_0 l_0 / D_0 \gg 1$



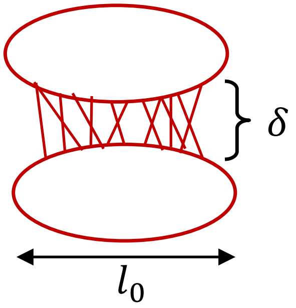
Isichenko '92

- Effective diffusivity? Transport
- Key Physics
  - irreversibility localized to inter-cell boundary
  - global transport hybrid { fast rotation in cell; slow diffusion in BL }

Estimating D_{eff} - Key Physics

- $D_{eff} \approx f_{active}\,[(\Delta x)^2/\Delta t]$

  fraction of space active in transport



$\delta \rightarrow$ boundary layer width
$l_0 \rightarrow$ cell size

- Boundary Layer
- $\delta^2 \sim D_0 \Delta t \sim D_0 l_0 / V_0$
- $\delta \sim (D_0 l_0 / V_0)^{1/2}$

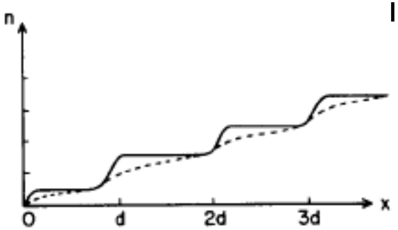
N.B. Not simple addition of processes

- $D_{eff} \sim \left(\frac{\delta}{l_0}\right)\left(\frac{l_0^2 V_0}{l_0}\right) \sim (D_0 D_{cell})^{1/2} \sim D_0 (P_e)^{1/2}$ intermediate result

What does the Profiles Look Likes?

- Have $D_{eff} \sim D_0 \, [Pe]^{1/2}$ but, { 2 time scales; averaging }

- Consider concentration of injected scalar



injection ⇦ ⇦ ⇦

c.f. "Steep transitions in the density exist between each cell."

→ Layering/Staircase !

→ Simple consequence of two rates

→ See Rosenbluth, et. al. '87, Shraiman '87 for detailed analysis

- Confirms that staircase arises in stationary array of passive eddys

What of Interest?

- Relevant to key question of "near marginal stability"
  - $V_0 l_0$ as turbulence
  - D_0 as neoclassical, ambient
- Contrived – Not so clear! – tracks conventional picture
  - Fixed cells – pinning at $\vec{k} \cdot \vec{B}_0 = 0$ surfaces
  - $\Delta r / L_\perp \ll 1$ – small ρ_*
  - barely overlapping cells $\leftrightarrow$ near marginality

  N.B. $D_{eff} \sim (D_0 D_{cell})^{1/2}$, not sum
- Layering as natural outcome in quasi-marginal state with 2 time scales
- Consider irregular staircases (c.f. Guo, et. al. '18) → Spin Glass ?

A Variation → Transverse Shear Flow

- To explore key role of time scale separation, add transverse shear

i.e. $\phi = \sin\left(\frac{\pi x}{d}\right)\sin\left(\frac{\pi \beta y}{d}\right) + \alpha\, \phi_{shear}$ $\quad \phi_{shear} = -\cos\left(\frac{mx}{2}\right)$

- 2 Peclet numbers

$$P_e = V_0 l_0 / D_0 \;\rightarrow\; d/D_0$$

$$P_{es} = V' / \left(\frac{D_0}{l_0^2}\right) \;\rightarrow\; \alpha m^2 d^2 / D_0$$

- Shear dispersion time scale:

$$\frac{1}{\tau_{SD}} = \left(\frac{D_0 V'^2}{l_0^2}\right)^{1/3} \Rightarrow \text{ effective mixing time}$$

Transverse Shear Flow, cont'd

- $\frac{1}{\tau_{SD}} > \frac{D_0}{l_0^2} \Rightarrow V' > \frac{D_0}{l_0^2}$ shear dispersion gives effective mixing rate faster than diffusion
- Then supercritical shear ➔ irreversible mixing outside inter-cell boundary layer

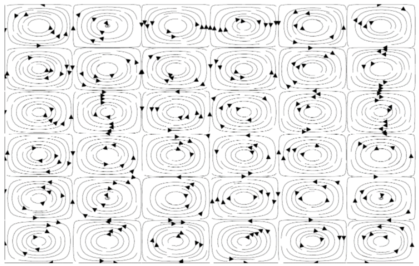
➔

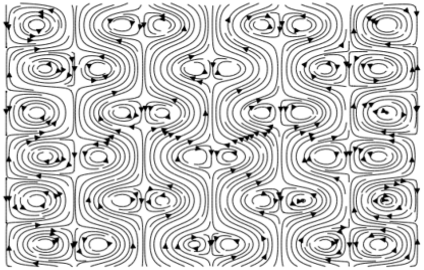
- Conjecture here that shear flow actually weakens staircase, by reducing slow-fast time scale ratio !

➔ Tests 'two scale' notion

Shear cont'd – System with Drive

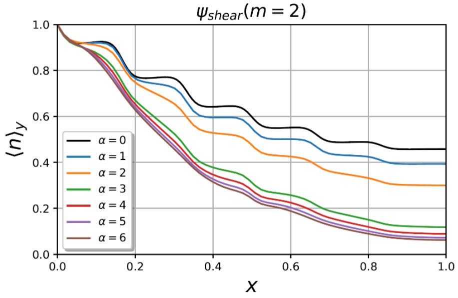
Flows





$\alpha = 4$

Concentration Profile (y-avg.)



- Staircase for $V' = 0$
- For $Pe_{sh} \gg 1$ corrugation decays
- Net particle confinement degrades

Supports conjecture !

Partial Conclusions II

- Staircase appears in stationary passive cellular array with diffusion at $Pe \gg 1$
- Simple consequence of two time scales, well separated, and their interplay in transport. Expulsion → transport barrier
- Relevant to "marginally overlapping cells", "near marginality"
- Enhanced shear mixing reduces τ_{slow}/τ_{fast} → degrades corrugation, staircase

Ongoing

- Noisey deposition
- Irregular cells
- Finite τ_c, statistical distribution

...

General Conclusions

- Two very simple system { Passive CH / Cell Array }
- Two different stationary passive scalar cell problems manifest staircase
- Two disparate time scales interact in each case

➔

- Layering is ubiquitous and to be expected in multi-scale problems!

Shameful Advertising

- For more on layering and staircases, from a broad perspective, see:
- https://online.kitp.ucsb.edu/online/staircase21/

  Kavli Inst. Program Staircase 21

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