Quasi-Geostrophic Fluids and Vlasov Plasmas: Parallels and Intersections

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Outline

- The Systems
- Review of the Quasilinear Problem – what’s important in Vlasov-Poisson?
- Potential Vorticity and the QG system
- Zonal flows and staircases
- Commonalities (Discussion)
- No conclusions

The Systems

- Vlasov-Poisson

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} + \frac{q}{m}E\frac{\partial f}{\partial v} = 0$$

f ≡ distribution function in x, v

$$\nabla^2\phi = -4\pi\int f dv \qquad f = \langle f\rangle + \delta f$$

➔ $\frac{df}{dt} = 0$ along trajectories

- $\partial_t\nabla^2\phi + \nabla\phi\times\hat{z}\cdot\nabla\nabla^2\phi = -\beta\partial_x\phi$

$$PV = \nabla^2\phi + \beta y$$

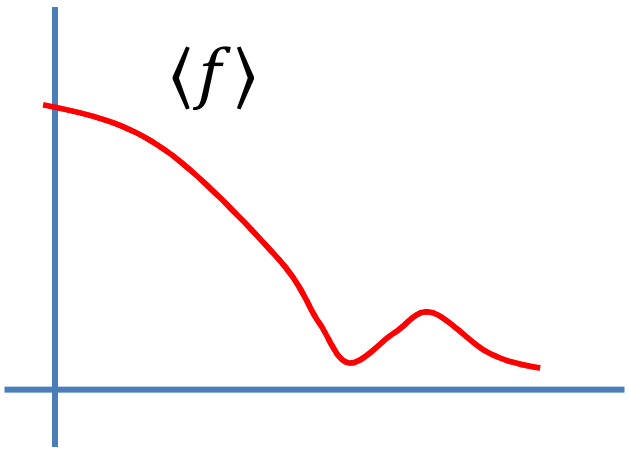
$\frac{d}{dt}(\nabla^2\phi + \beta y) = 0$ along trajectories

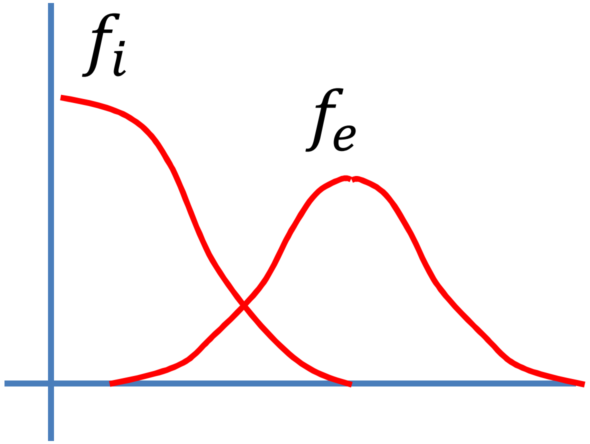
drag, diffusion

Vlasov-Poisson Revisited

Key Problem – QL for Vlasov Plasma – Validity?

- Good beginnings: Vedenov, Velikov, Sagdeev; Drummond, Pines
  - 1D Vlasov evolution / relaxation of B-O-T, CDIA





Inputs

- Landau theory
- Stochasticity
- Radiative transfer theory
- Mean field for $\langle N \rangle$

  - QL system, from mean field approach with linear response

$$\epsilon(k,\omega) = 0, \quad \partial_t \langle f \rangle = \frac{\partial}{\partial v} D \frac{\partial \langle f \rangle}{\partial v} \qquad \partial_t |E_k|^2 = 2\gamma_k |E_k|^2 \qquad D = D(|E|^2)$$

- Key:

  - $D = \frac{q^2}{m^2} \sum_k |E_k|^2 \frac{|\gamma_k|}{(\omega - kv)^2 + |\gamma_k|^2}$
  - Resonant → $\pi\delta(\omega - kv)$ → irreversible
  - Non-resonant → $|\gamma_k| / \omega_k^2$ → reversible / ‘fake’
  - Non-resonant diffusion for stationary turbulence is problematic. Energetics?
  - Coarse graining implicit in $\langle \, \rangle$
  - First derivation via RPA, ultimately particle stochasticity is fundamental
  - Finite width resonance → local PV mixing

- Central elements/orderings:
  - resonant diffusion, irreversibility:
    - “chaos” ←→ coarse graining
    - Island overlap at resonances: $\frac{\omega}{k_{i+i}} - \frac{\omega}{k_i} \leq \sqrt{q\phi\,/m}$
  - linear response?:
    - $\tau_{ac} < \tau_{tr}\,,\ \tau_{decorr}\ ,\ \ \gamma_k$
    - $\tau_{ac}^{-1} = \left|\frac{d\omega}{dk} - \frac{\omega}{k}\right|\ |\Delta k|$ → correlation time of wave-particle resonance
    - $\tau_{tr}^{-1} = k\,\sqrt{q\phi\,/m}$ → particle bounce time in pattern
    - $\tau_{decorr}^{-1}\ = (k^2 D)^{1/3}$ → particle decorrelation rate (cf. Dupree ’66)

- QLT is Kubo # < 1 theory

i.e. $\frac{q}{m}\tilde{E}\tau_{ac}/\Delta v_T = \Delta v_T\, k\, \tau_{ac} < 1$

but often pushed to Ku ~ 1 !

- QLT assumes:
  - all fluctuations are eigenmodes (i.e. neglect mode coupling)?
  - all $\delta f \sim \tilde{E}\, \partial\langle f\rangle/\partial v$?

    (resemble $\delta B \sim \tilde{v}\langle B\rangle$ in MF dynamo theory)

- Energetics → 2 component description
  - Resonant Particles vs Waves

$$\partial_t(RPKED) + \partial_t(W\ ED) = 0$$

or

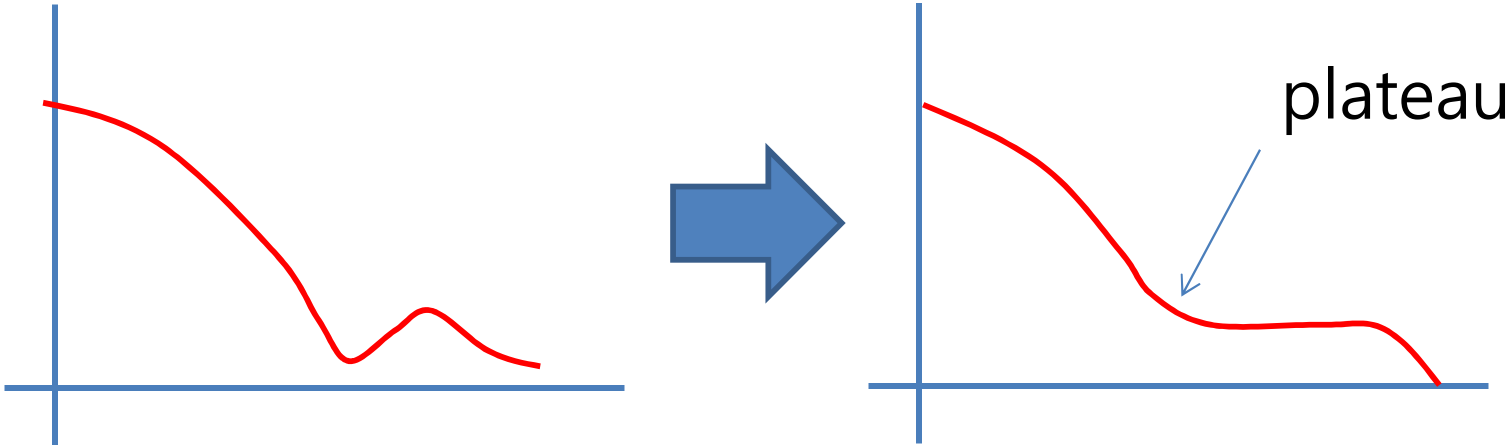
  - Particles vs Fields

$$\partial_t(PKED) + \partial_t(FED) = 0$$

  - Species coupled via waves
  - Issues: how describe stationary state with RP drive?

i.e. $D_R \left(\frac{\partial\langle f\rangle}{\partial v}\right)^2 = d_{col} \ \langle\left(\frac{\partial \delta f}{\partial v}\right)^2\rangle$, ala' Zeldovich

- Outcome:
  - B-O-T: Plateau formation



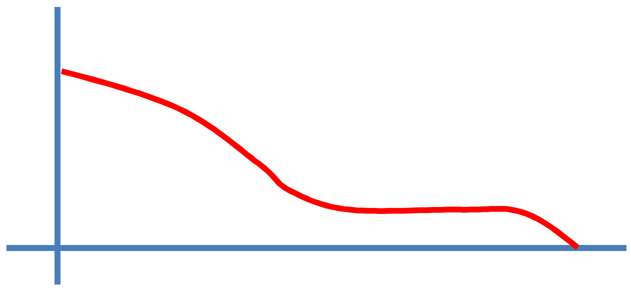
  - prediction for $|\tilde{E}_{sat}|^2 / 4\pi nT$ when plateau formed
  - CDIA:
    - wave driven momentum transfer e->i
    - anomalous resistivity model (quasi-marginality)

- Why Plateau?
  - In collisionless, un-driven system, need at stationarity:

$$\int dv\, D_R (\partial\langle f\rangle/\partial v)^2 = 0$$

  - So either: (collisions: RHS → $d_{col} \ \langle \left(\frac{\partial\delta f}{\partial v}\right)^2 \rangle$)

i) $\partial\langle f\rangle/\partial v = 0$, where $D(v) \neq 0$ on interval → plateau with finite amplitude waves



ii) Or $D_R = 0$ → fluctuation decay everywhere, $\gamma_k < 0$

  - Sub-overlap → velocity space staircase

Is this story correct?

- TWT experiment – last time
- Granulations, phase space holes, etc.

  ➔ subcritical growth ?!
- See M. Lesur, this meeting

Potential Vorticity,

Quasi-Geostrophics

and Hasegawa-Mina

The System Fundamentals: $R_0 \ll 1$ Fluids $(\Omega \leftrightarrow \Omega_i)$

- Kelvin's Theorem for rotating system

$$\omega \to \omega + 2\mathbf{\Omega}$$

relative planetary

→

$$\oint \mathbf{v} \cdot d\mathbf{l} = \int d\mathbf{a} \cdot (\omega + 2\Omega) \equiv C$$

$$\dot{C} = 0$$

- $Ro = V/(2\Omega L) \ll 1 \quad \to \quad \mathbf{V} \cong -\nabla_\perp p \times \hat{z}/(2\Omega)$ geostrophic balance

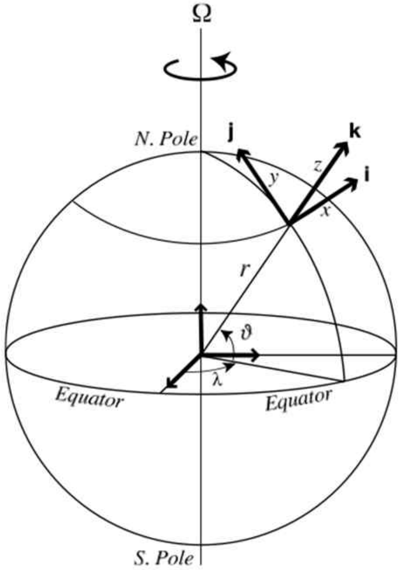
→ 2D dynamics

- Displacement on beta plane

$$\dot{C} = 0 \quad \to \quad \frac{d}{dt}\omega \cong -\frac{2\Omega}{A}\sin\theta_0\frac{dA}{dt}$$

$$= -2\Omega\frac{d\theta}{dt} = -\beta V_y$$

$$\omega = \nabla^2\phi \qquad \beta = 2\Omega\sin\theta_0/R$$



cf. Lynden-Bell, Vlasov

Fundamentals II

- Q.G. equation $\frac{d}{dt}(\omega + \beta y) = 0$

- Locally Conserved PV $q = \omega + \beta y$

n.b. topography

$q = \omega / H + \beta y$

  - Latitudinal displacement → change in relative vorticity

  - Linear consequence → Rossby Wave

$$\omega = -\beta k_x / k^2$$

$\omega = 0$ → zonal flow

observe: $v_{g,y} = 2\beta \underline{k_x k_y} / (k^2)^2$

→ Rossby wave intimately connected to momentum transport

- Latitudinal PV Flux → circulation

→ Isn’t this Talk re: Plasma?

→ 2 Simple Models
a.) Hasegawa-Wakatani (collisional drift inst.)
b.) Hasegawa-Mima (DW)

a.) $\mathbf{V} = \frac{c}{B}\hat{z} \times \nabla\phi + \underset{\rightarrow m_s}{\mathbf{V}_{pol}}$

$$L > \lambda_D \rightarrow \nabla \cdot \mathbf{J} = 0 \rightarrow \nabla_\perp \cdot \mathbf{J}_\perp = -\nabla_\parallel J_\parallel$$

$$J_\perp = n|e|V_{pol}^{(i)}$$

$$J_\parallel : \eta J_\parallel = -\cancel{(1/c)\partial_t A_\parallel} - \nabla_\parallel \phi + \nabla_\parallel p_e$$

e.s.

b.) $dn_e/dt = 0$

$$\rightarrow \quad \frac{dn_e}{dt} + \frac{\nabla_\parallel J_\parallel}{-n_0|e|} = 0$$

n.b.

MHD: $\partial_t A_\parallel$ v.s. $\nabla_\parallel \phi$

DW: $\nabla_\parallel p_e$ v.s. $\nabla_\parallel \phi$

So H-W

$$\rho_s^2 \frac{d}{dt}\nabla^2\hat{\phi} = -D_{\|}\nabla_{\|}^2(\hat{\phi} - \hat{n}/n_0) + \nu\nabla^2\nabla^2\hat{\phi}$$

$$\frac{d}{dt}n - D_0\nabla^2\hat{n} = -D_{\|}\nabla_{\|}^2(\hat{\phi} - \hat{n}/n_0)$$

$D_{\|}k_{\|}^2/\omega$

is key parameter

$\rightarrow \langle\tilde{v}_r\tilde{n}\rangle \neq 0$

and instability

b.) $D_{\|}k_{\|}^2/\omega \gg 1 \rightarrow \hat{n}/n_0 \sim e\hat{\phi}/T_e \qquad (m, n \neq 0)$

$$\frac{d}{dt}(\phi - \rho_s^2\nabla^2\phi) + v_*\partial_y\phi = 0$$

→ H-M

n.b. $\mathrm{PV} = \phi - \rho_s^2\nabla^2\phi + \ln n_0(x) \qquad \frac{d}{dt}(\mathrm{PV}) = 0$

An infinity of technical models follows …

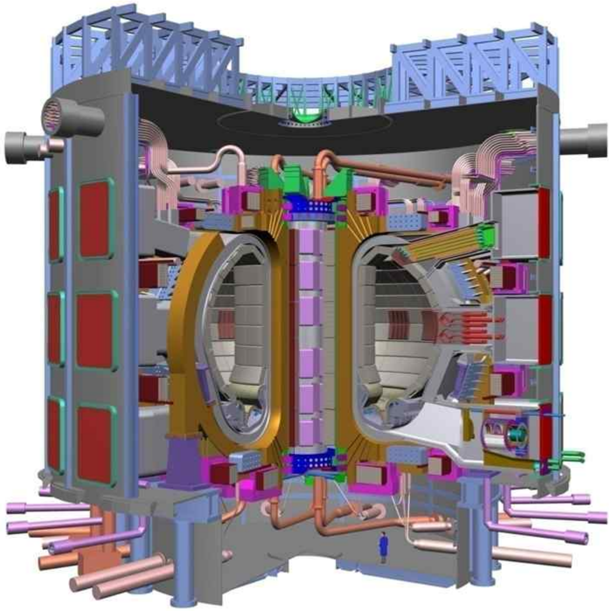
Preamble Ⅰ

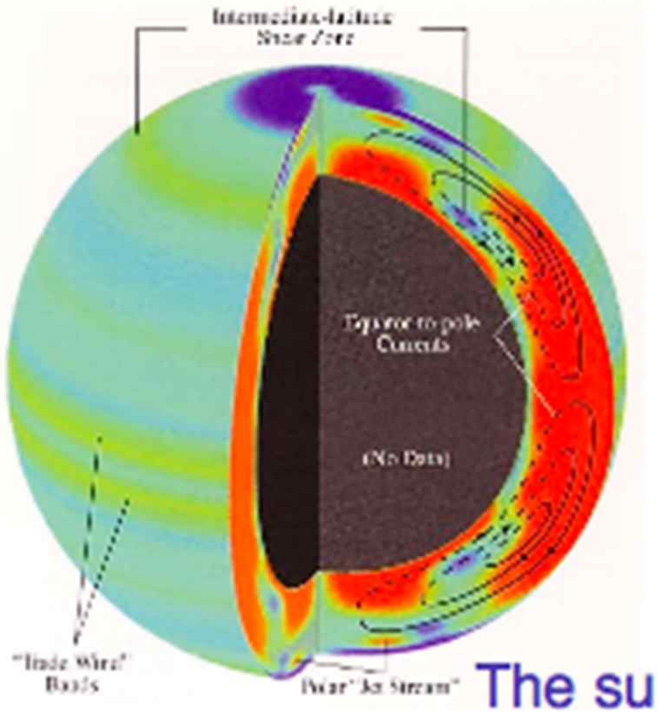
- Zonal Flows Ubiquitous for:

  ~ 2D fluids / plasmas $R_0 < 1$

  Rotation $\vec{\Omega}$, Magnetization $\vec{B}_0$, Stratification

  Ex: MFE devices, giant planets, stars…





The sun

Preamble II

- What is a Zonal Flow?
  - – *n = 0* potential mode; *m = 0* (ZFZF), with possible sideband (GAM)
  - – toroidally, poloidally symmetric *ExB* shear flow
- Why are Z.F.'s important?
  - – Zonal flows are secondary (nonlinearly driven):
    - modes of minimal inertia (Hasegawa et. al.; Sagdeev, et. al. '78)
    - modes of minimal damping (Rosenbluth, Hinton '98)
    - drive zero transport (*n = 0*)
  - – natural predators to feed off and retain energy released by gradient-driven microturbulence

Zonal Flows I

- Fundamental Idea:
  - Potential vorticity transport + 1 direction of translation symmetry
    → Zonal flow in magnetized plasma / QG fluid
  - Kelvin's theorem is ultimate foundation

- G.C. ambipolarity breaking → polarization charge flux → Reynolds force
  - Polarization charge ➡ $-\boxed{\rho^2}\nabla^2\phi = \boxed{n_{i,GC}(\phi) - n_e(\phi)}$

    polarization length scale ↵ ↳ *ion GC* ↳ *electron density*
  - so $\Gamma_{i,GC} \neq \Gamma_e$ ➡ $\rho^2\left\langle \tilde{v}_{rE}\nabla_\perp^2\tilde{\phi}\right\rangle \neq 0$ ⬌ 'PV transport'

    ↳ *polarization flux* → What sets cross-phase?
  - If 1 direction of symmetry (or near symmetry):

$$-\rho^2\left\langle \tilde{v}_{rE}\nabla_\perp^2\tilde{\phi}\right\rangle = -\partial_r\left\langle \tilde{v}_{rE}\tilde{v}_{\perp E}\right\rangle$$ (Taylor, 1915)

$-\partial_r\left\langle \tilde{v}_{rE}\tilde{v}_{\perp E}\right\rangle$ ➡ Reynolds force ➡ Flow

Zonal Flows II

- Potential vorticity transport and momentum balance
  - Example: Simplest interesting system → Hasegawa-Wakatani
    - Vorticity: $\frac{d}{dt}\nabla^2\phi = -D_{\|}\nabla_{\|}^2(\phi - n) + D_0\nabla^2\nabla^2\phi$
    - Density: $\frac{dn}{dt} = -D_{\|}\nabla_{\|}^2(\phi - n) + D_0\nabla^2 n$

$$\left\{\begin{array}{ll} D_0 & \text{classical, feeble} \\ \Pr = 1 & \text{for simplicity} \end{array}\right.$$

  - Locally advected PV: $q = n - \nabla\phi^2$
    - PV: charge density $\left\{\begin{array}{l} n \rightarrow \text{guiding centers} \\ -\nabla\phi^2 \rightarrow \text{polarization} \end{array}\right.$
    - conserved on trajectories in inviscid theory $\boxed{dq/dt=0}$
    - PV conservation → $\left.\begin{array}{l}\text{Freezing-in law} \\ \text{Kelvin's theorem}\end{array}\right\} \rightarrow$ Dynamical constraint

Zonal Flow II, cont'd

- Potential Enstrophy (P.E.) balance

$$d\langle q^2\rangle / dt = 0 \qquad \langle\ \rangle \rightarrow \text{coarse graining}$$

$$\text{LHS} \Rightarrow \frac{d}{dt}\langle\tilde{q}^2\rangle \equiv \partial_t\langle\tilde{q}^2\rangle + \partial_r\overset{\substack{\text{flux}\\ \downarrow}}{\langle\tilde{V}_r\tilde{q}^2\rangle} + D_0\overset{\substack{\text{dissipation}\\ \downarrow}}{\langle(\nabla\tilde{q})^2\rangle}$$

$\text{RHS} \Rightarrow \text{P.E. evolution} - \langle\tilde{V}_r\tilde{q}\rangle\langle q\rangle' \Rightarrow$ P.E. Production by PV mixing / flux

- PV flux: $\langle\tilde{V}_r\tilde{q}\rangle = \langle\tilde{V}_r\tilde{n}\rangle - \langle\tilde{V}_r\nabla^2\tilde{\phi}\rangle$; but : $\langle\tilde{V}_r\nabla^2\tilde{\phi}\rangle = \partial_r\langle\tilde{V}_r\tilde{V}_\theta\rangle$

∴ P.E. production directly couples driving transport and flow drive

- Fundamental Stationarity Relation for Vorticity flux

$$\underset{\substack{\uparrow\\ \text{Reynolds force}}}{\langle\tilde{V}_r\nabla^2\tilde{\phi}\rangle} = \underset{\substack{\uparrow\\ \text{Relaxation}}}{\langle\tilde{V}_r\tilde{n}\rangle} + (\delta_t\underset{\substack{\uparrow\\ \text{Local PE decrement}}}{\langle\tilde{q}^2\rangle}))/\langle q\rangle'$$

∴ Reynolds force locked to driving flux and P.E. decrement; transcends quasilinear theory

Zonal Flows III

- Momentum Theorem (Charney, Drazin 1960, et. seq. P.D. et. al. ‘08)

$$\partial_t \left\{ (GWMD) + \langle V_\theta \rangle \right\} = -\langle \tilde{V}_r \tilde{n} \rangle - \delta_t \langle \tilde{q}^2 \rangle / \langle q \rangle' - \nu \langle V_\theta \rangle$$

drag

driving flux

Local P.E. decrement

GWMD = Generalized Wave Momentum Density; $\langle \tilde{q}^2 \rangle / \langle q \rangle'$

- What Does it Mean? “Non-Acceleration Theorem”:

$$\partial_t \left\{ (GWMD) + \langle V_\theta \rangle \right\} = -\langle \tilde{V}_r \tilde{n} \rangle - \delta_t \langle \tilde{q}^2 \rangle / \langle q \rangle' - \nu \langle V_\theta \rangle$$

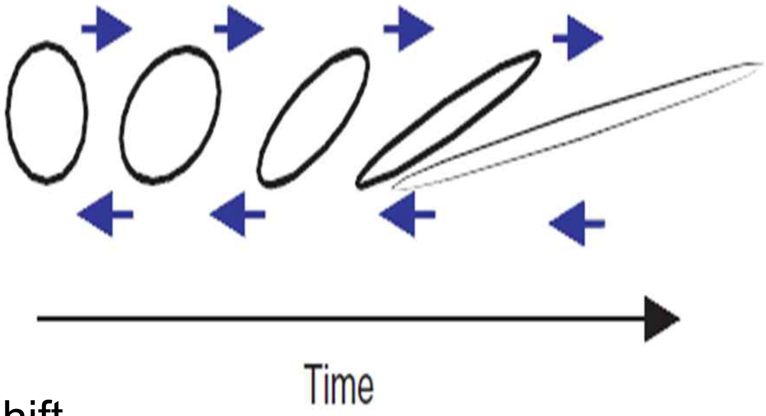
  - Absent $\langle \tilde{V}_r \tilde{n} \rangle$ driving flux; $\delta_t \langle \tilde{q}^2 \rangle$ — local potential enstrophy decrement

    → cannot $\left\{ \begin{matrix} \text{accelerate} \\ \text{maintain} \end{matrix} \right\}$ Z.F. with stationary fluctuations!

- Fundamental constraint on models of stationary zonal flows! ↔ need explicit connection to relaxation, dissipation

Shearing I

- Coherent shearing: (Kelvin, G.I. Taylor, Dupree'66, BDT'90)
  - radial scattering + $\langle V_E \rangle'$ → hybrid decorrelation
  - $k_r^2 D_\perp \rightarrow (k_\theta^2 \langle V_E \rangle'^2 D_\perp / 3)^{1/3} = 1/\tau_c$
  - shaping, flux compression: Hahm, Burrell '94



- Other shearing effects (linear):

Response shift
and dispersion

  - spatial resonance dispersion: $\omega - k_\| v_\| \Rightarrow \omega - k_\| v_\| - k_\theta \langle V_E \rangle' (r - r_0)$
  - differential response rotation → especially for kinetic curvature effects

→ N.B. Caveat: Modes can adjust to weaken effect of external shear

(Carreras, et. al. '92; Scott '92)

Shearing II

- Zonal Shears: Wave kinetics (Zakharov et. al.; P.D. et. al. '98, et. seq.)

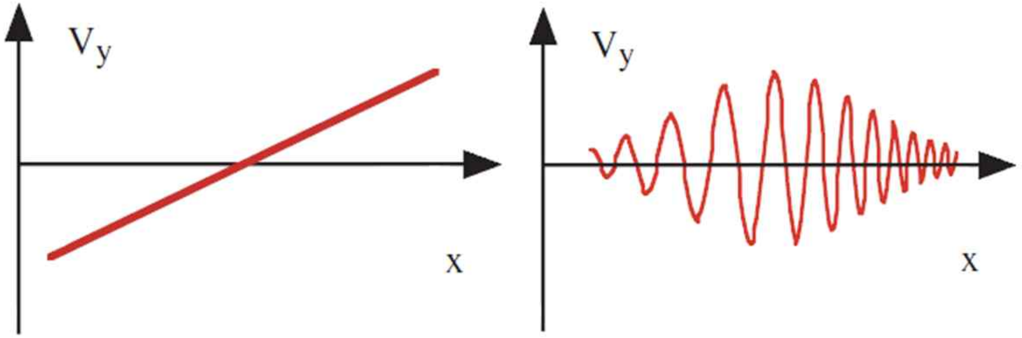
  Coherent interaction approach (L. Chen et. al.)

- $dk_r / dt = -\partial(\omega + k_\theta V_E) / \partial r\,;\ V_E = \langle V_E \rangle + \widetilde{V}_E$

  Mean shearing $: k_r = k_r^{(0)} - k_\theta V_E' \tau$

  Zonal Random shearing $: \langle \delta k_r^2 \rangle = D_k \tau$

  $$D_k = \sum_q k_\theta^2 \left| \widetilde{V}_{E,q}' \right|^2 \tau_{k,q}$$



– Wave ray chaos (not shear RPA) underlies $D_k \rightarrow$ induced diffusion

– Induces wave packet dispersion

– Applicable to ZFs and GAMs

- Mean Field Wave Kinetics

$$\frac{\partial N}{\partial t} + (\vec{V}_{gr} + \vec{V}) \cdot \nabla N - \frac{\partial}{\partial r}(\omega + k_\theta V_E) \cdot \frac{\partial N}{\partial \vec{k}} = \gamma_{\vec{k}} N - C\{N\}$$

$$\Rightarrow \boxed{\frac{\partial}{\partial t}\langle N \rangle - \frac{\partial}{\partial k_r} D_k \frac{\partial}{\partial k_r}\langle N \rangle = \gamma_{\vec{k}} \langle N \rangle - \langle C\{N\} \rangle}$$

Zonal shearing

Shearing III

- Energetics: Books Balance for Reynolds Stress-Driven Flows!
- Fluctuation Energy Evolution – Z.F. shearing

$$\int d\vec{k}\,\omega\left(\frac{\partial}{\partial t}\langle N\rangle-\frac{\partial}{\partial k_r}D_k\frac{\partial}{\partial k_r}\langle N\rangle\right)\Rightarrow\frac{\partial}{\partial t}\langle\varepsilon\rangle=-\int d\vec{k}\,V_{gr}(\vec{k})D_{\vec{k}}\frac{\partial}{\partial k_r}\langle N\rangle \qquad V_{gr}=\frac{-2k_rk_\theta V_*\rho_s^2}{\left(1+k_\perp^2\rho_s^2\right)^2}$$

Point: For $d\langle\Omega\rangle/dk_r<0$, Z.F. shearing damps wave energy

- Fate of the Energy: Reynolds work on Zonal Flow

Modulational Instability

$$\partial_t\delta V_\theta+\partial\left(\delta\left\langle\tilde{V}_r\tilde{V}_\theta\right\rangle\right)/\partial r=-\gamma\delta V_\theta$$

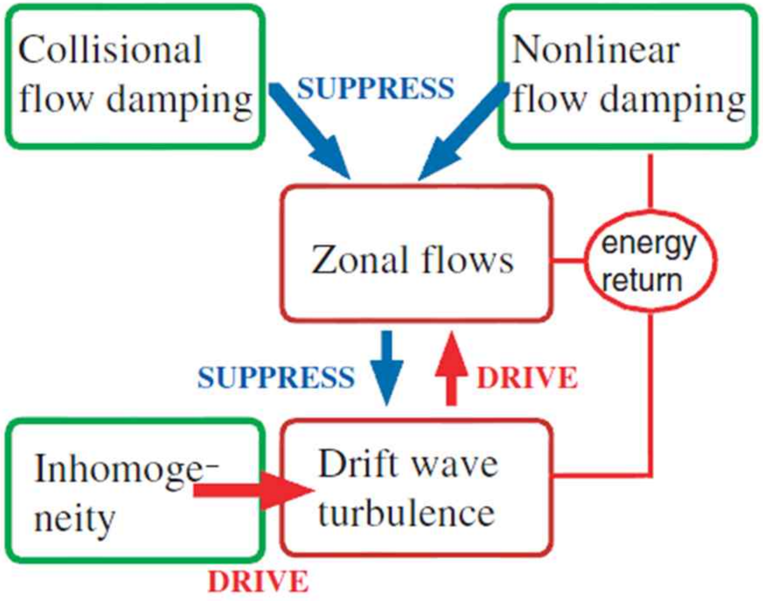
$$\delta\left\langle\tilde{V}_r\tilde{V}_\theta\right\rangle\sim\frac{k_rk_\theta\delta\Omega}{(1+k_\perp^2\rho_s^2)^2}$$

N.B.: Wave decorrelation essential:
Equivalent to PV transport
(c.f. Gurcan et. al. 2010)

- Bottom Line:
  - Z.F. growth due to shearing of waves
  - “Reynolds work” and “flow shearing” as relabeling → books balance
  - Z.F. damping emerges as critical; MNR ‘97

Feedback Loops I

- Closing the loop of shearing and Reynolds work



- Spectral ‘Predator-Prey’ equations

Prey → Drift waves, <*N*>

$$\frac{\partial}{\partial t}\langle N\rangle-\frac{\partial}{\partial k_r}D_k\frac{\partial}{\partial k_r}\langle N\rangle=\gamma_k\langle N\rangle-\frac{\Delta\omega_k}{N_0}\langle N\rangle^2$$

Predator → Zonal flow, $|\phi_q|^2$

$$\frac{\partial}{\partial t}|\phi_q|^2=\Gamma_q\left[\frac{\partial\langle N\rangle}{\partial k_r}\right]|\phi_q|^2-\gamma_d|\phi_q|^2-\gamma_{NL}[|\phi_q|^2]|\phi_q|^2$$

Structures:

- Inhomogeneous PV mixing

  ➔ Staircases

- Overlap and the gradient domino effect

  ➔ Avalanches

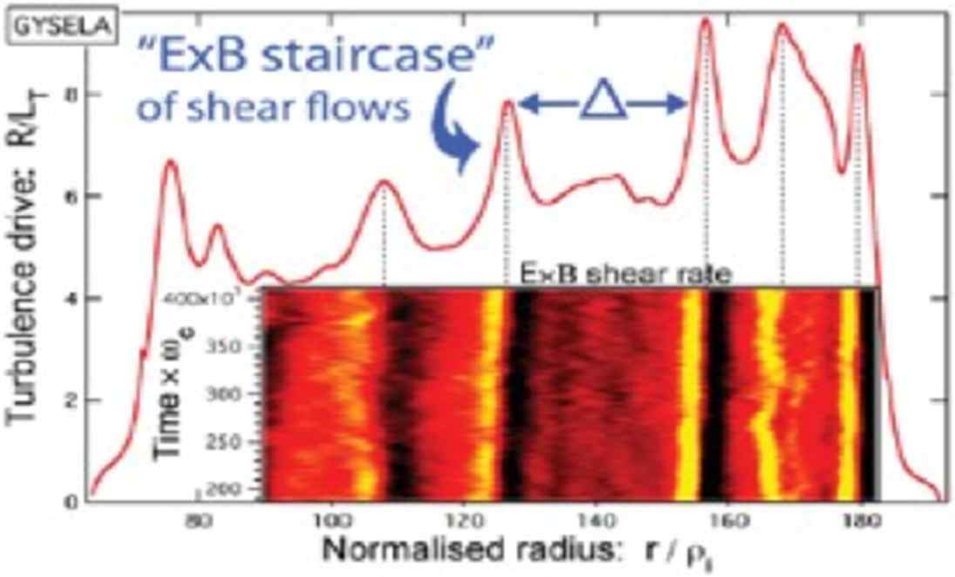
Motivation: ExB staircase formation

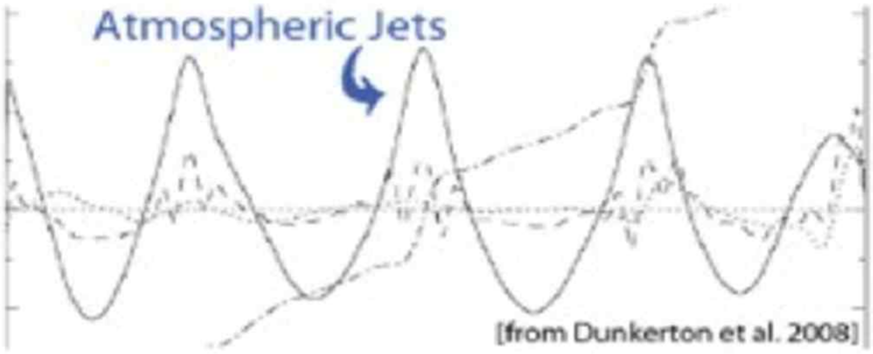
- ExB flows often observed to self-organize in magnetized plasmas

  eg. mean sheared flows, zonal flows, ...

- \`ExB staircase' is observed to form

(G. Dif-Pradalier, P.D. et al. Phys. Rev. E. '10)





[from Dunkerton et al. 2008]

- flux driven, full f simulation
- Quasi-regular pattern of shear layers and profile corrugations
- Region of the extent $\Delta \gg \Delta_c$ interspersed by temp. corrugation/ExB jets

  → ExB staircases
- so-named after the analogy to PV staircases and atmospheric jets
- Step spacing → avalanche outer-scale

- Interesting as:
  - Clear scale selection
  - Clear link of:
  - ZF scale ←→ avalanche scale → corrugation

But:

  - Systematic scans lacking
  - Somewhat difficult to capture

- Need a <u>MODEL</u>

The Hasegawa-Wakatani Staircase:

Profile Structure:

From Mesoscopics → Macroscopics

H-W Drift wave model – Fundamental prototype

- Hasegawa-Wakatani : simplest model incorporating instability

$$V=\frac{c}{B}\hat{z}\times\nabla\phi+V_{pol}$$

$$J_{\perp}=n|e|V^{i}_{pol} \qquad \eta J_{\parallel}=-\nabla_{\parallel}\phi+\nabla_{\parallel}p_{e}$$

$$\nabla_{\perp}\cdot J_{\perp}+\nabla_{\parallel}J_{\parallel}=0$$ → vorticity: $$\rho_s^2\frac{d}{dt}\nabla^2\phi=-D_{\parallel}\nabla_{\parallel}^2(\phi-n)+\nu\nabla^2\nabla^2\phi$$

$$\frac{dn_e}{dt}+\frac{\nabla_{\parallel}J_{\parallel}}{-n_0|e|}=0$$ → density: $$\frac{d}{dt}n=-D_{\parallel}\nabla_{\parallel}^2(\phi-n)+D_0\nabla^2 n$$

→ PV conservation in inviscid theory $$\frac{d}{dt}\left(n-\nabla^2\phi\right)=0$$

→ PV flux = particle flux + vorticity flux

→ zonal flow being a counterpart of particle flux

QL: $$\frac{\partial}{\partial t}\langle n\rangle=-\frac{\partial}{\partial r}\langle\tilde{\upsilon}_r\tilde{n}\rangle$$

→? $$\frac{\partial}{\partial t}\langle\nabla^2\phi\rangle=-\frac{\partial}{\partial r}\langle\tilde{\upsilon}_r\nabla^2\tilde{\phi}\rangle$$

$$=-\frac{\partial^2}{\partial r^2}\langle\tilde{\upsilon}_r\tilde{\upsilon}_\theta\rangle$$

- Hasegawa-Mima ($D_{\parallel}k_{\parallel}^2/\omega>>1 \;\rightarrow\; n\sim\phi$)

$$\frac{d}{dt}\left(\phi-\rho_s^2\nabla^2\phi\right)+\upsilon_*\partial_y\phi=0$$

The Reduced 1D Model

Reduced system of evolution Eqs. is obtained from HW system for DW turbulence.

Variables:

$u = \partial_x V_y$ Zonal shearing field

Reduced density: $\log(N/N_0) = \boxed{n(x,t)} + \tilde{n}(x,y,t),$ **Vorticity:** $\rho_s^2 \nabla_\perp^2 \left(e\varphi/T_e\right) = \boxed{u(x,t)} + \tilde{u}(x,y,t)$

Potential Vorticity (PV): $q = n - u,$ **Turbulent Potential Enstrophy (PE):** $\varepsilon = \frac{1}{2}\left\langle (\tilde{n} - \tilde{u})^2 \right\rangle$

Two components

Mean field equations:

density $\partial_t n = -\partial_x \Gamma_n + \partial_x [D_c \partial_x n],$ $\Gamma_n = \left\langle \tilde{v}_x \tilde{n} \right\rangle = -D_n \partial_x n \longrightarrow$ **Reflect instability**

$\text{Taylor ID}: \Pi_u = \left\langle \tilde{v}_x \tilde{u} \right\rangle = \partial_x \left\langle \tilde{v}_x \tilde{v}_y \right\rangle$

vorticity $\partial_t u = -\partial_x \Pi_u + \partial_x [\mu_c \partial_x u],$ $\Pi_u = \left\langle \tilde{v}_x \tilde{u} \right\rangle = (\chi - D_n)\partial_x n - \boxed{\chi}\partial_x u$

Residual vort. flux **Turb. viscosity**

Turbulence evolution: (Potential Enstrophy)

From closure

$$\partial_t \varepsilon = \partial_x [D_\varepsilon \partial_x \varepsilon] - (\Gamma_n - \Gamma_u)[\partial_x (n - u)] - \varepsilon_c^{-1} \varepsilon^{3/2} + P$$

Turbulence spreading · Internal production · dissipation · External production $\sim \gamma\varepsilon$

Two fluxes Γ_n, Γ_u set model !

What is new in this model?

○ **In this model PE conservation is a central feature.**

○**Mixing of Potential Vorticity (PV) is the fundamental effect regulating the interaction between turbulence and mean fields. Mixing <u>inhomogeneous</u>**

○**Dimensional and physical arguments used to obtain functional forms for the turbulent diffusion coefficients. From the QL relation for HW system we obtain**

$$D_n \cong l^2 \frac{\varepsilon}{\alpha} \qquad \chi \cong c_\chi l^2 \frac{\varepsilon}{\sqrt{\alpha^2 + a_u u^2}}$$

* l **<u>Dynamic mixing length</u>** → **Rhines scale sets**

α **Parallel diffusion rate**

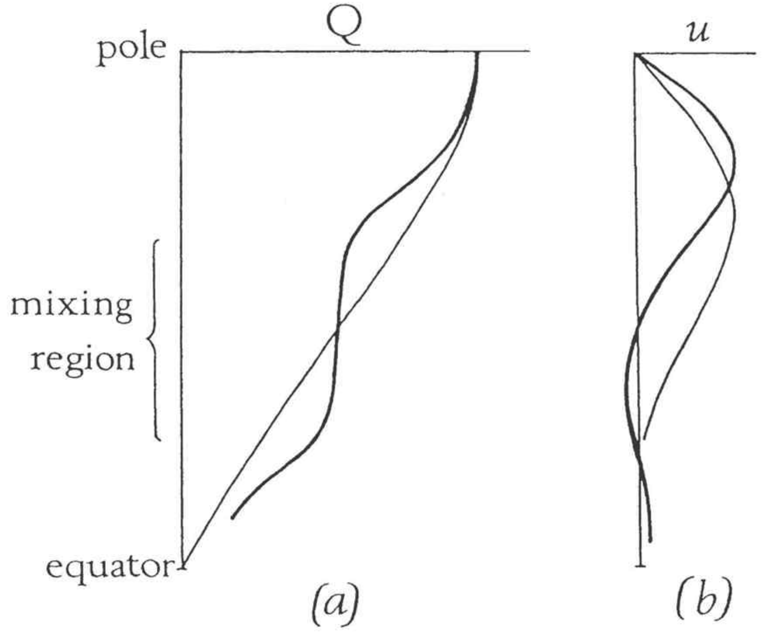
○***Inhomogeneous mixing of PV results in the sharpening of density and vorticity gradients in some regions and weakening them in other regions, leading to shear lattice and density staircase formation.***

Jet sharpening in stratosphere, resulting from inhomogeneous mixing of PV. (McIntyre 1986)

PV $Q = \nabla^2 \psi + \beta y$

Relative vorticity

Planetary vorticity



Perspective on (Rhines) Scale

- Note: $l^2 = \frac{1}{1+1/l_{Rh}^2} \rightarrow \frac{1}{1+\langle q\rangle'^2/\epsilon}$ $(l_f \sim 1)$
- Reminiscent of weak turbulence perspective:

$$D = D_{pv} = \sum_{\vec{k}} \frac{\langle \tilde{V}^2\rangle \Delta\omega_{\vec{k}}}{\omega_{\vec{k}}^2 + \Delta\omega_{\vec{k}}^2}$$

$$\omega_{\vec{k}} = -k_x \langle q\rangle'/k^2$$

$$\Delta\omega_{\vec{k}} \approx k\tilde{V}_{\vec{k}}$$

Ala' Dupree'67:

$$D_{pv} \approx \frac{1}{k^2}\left(\sum_{\vec{k}} k^2 \langle \tilde{V}^2\rangle_{\vec{k}} - \frac{k_x^2\left(\langle q\rangle'\right)^2}{(k^2)^2}\right)^{1/2}$$

Steeper $\langle q\rangle'$ quenches diffusion → mixing reduced via PV gradient feedback

$$D_{pv} \approx \frac{l_0^2\ \epsilon^{\frac{1}{2}}}{1+\frac{l_0^2}{\epsilon}(\langle q\rangle')^2} \qquad \leftarrow$$

- ω vs $\Delta\omega$ dependence gives D_{pv} roll-over with steepening
- Rhines scale appears naturally, in feedback strength → roll over scale
- Recovers effectively same model

Physics:

① “Rossby wave elasticity’ (MM) → steeper $\langle q\rangle'$ → stronger memory (i.e. more ‘waves’ vs turbulence)

→ ② Distinct from shear suppression → interesting to dis-entangle

Staircase structure

Snapshots of evolving profiles at t=1 (non-dimensional time)

Initial conditions: $n = g_0(1-x), \quad u = 0, \quad \varepsilon = \varepsilon_0$

Boundary conditions: $n(0,t) = g_0,\ n(1,t) = 0; \quad u(0,1;t) = 0; \quad \partial_x \varepsilon(0,1;t) = 0$

Structures:

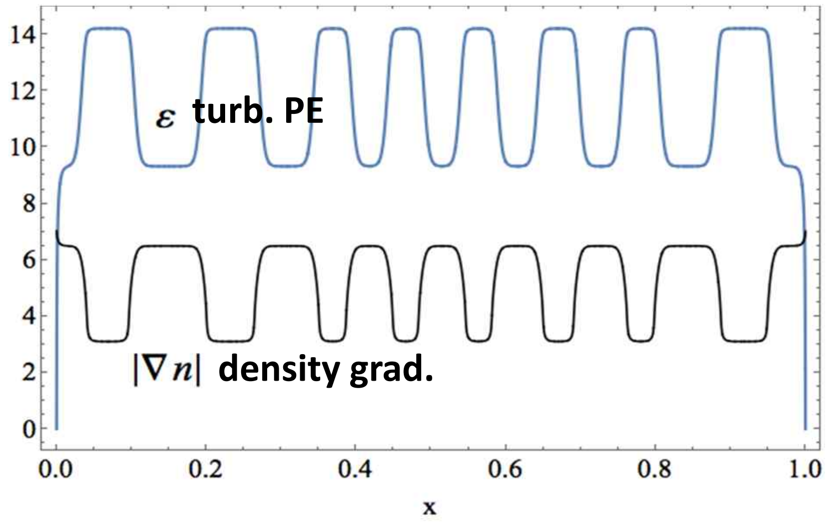
○**Staircase in density profile:**

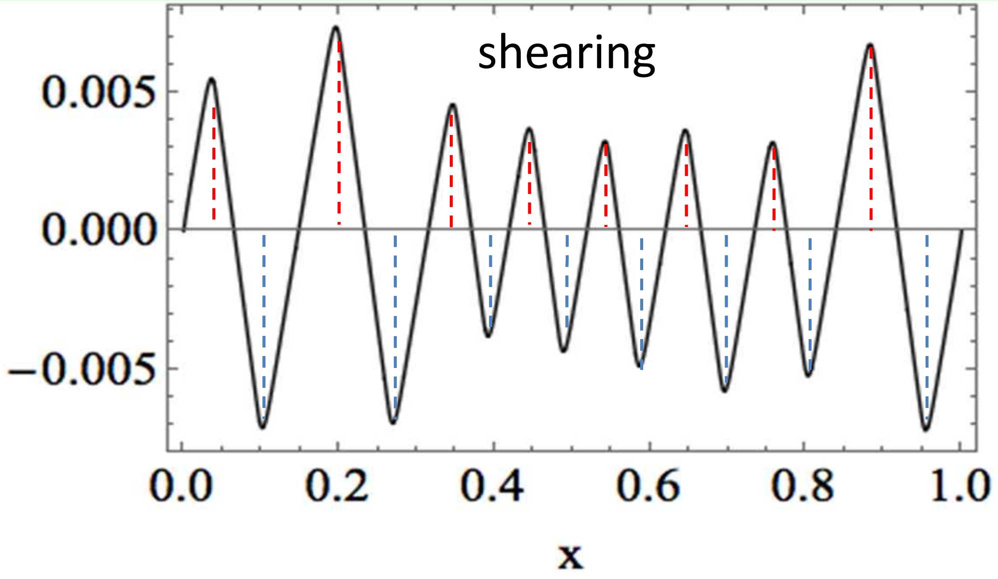
jumps → regions of steepening

steps → regions of flattening

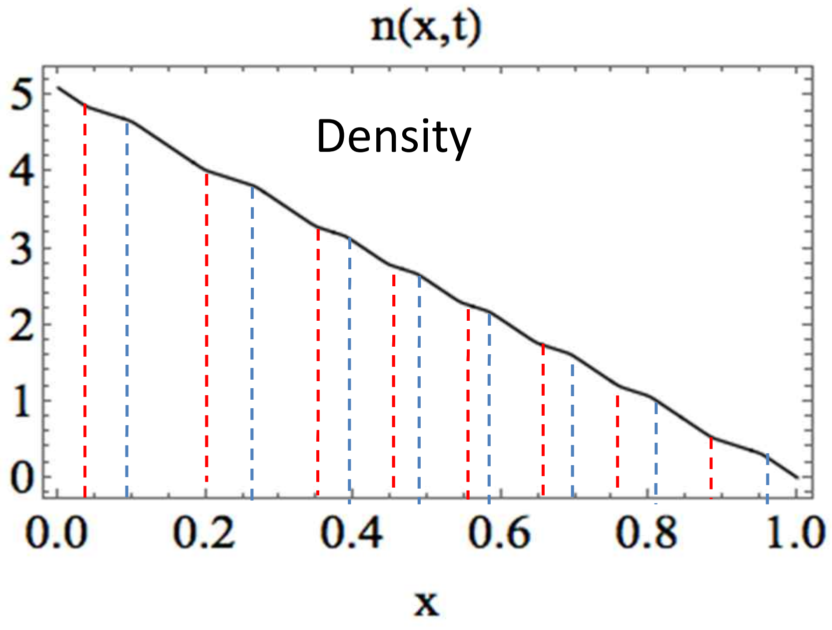
○**At the jump locations, turbulent PE is suppressed.**

○**At the jump locations, vorticity gradient is positive**

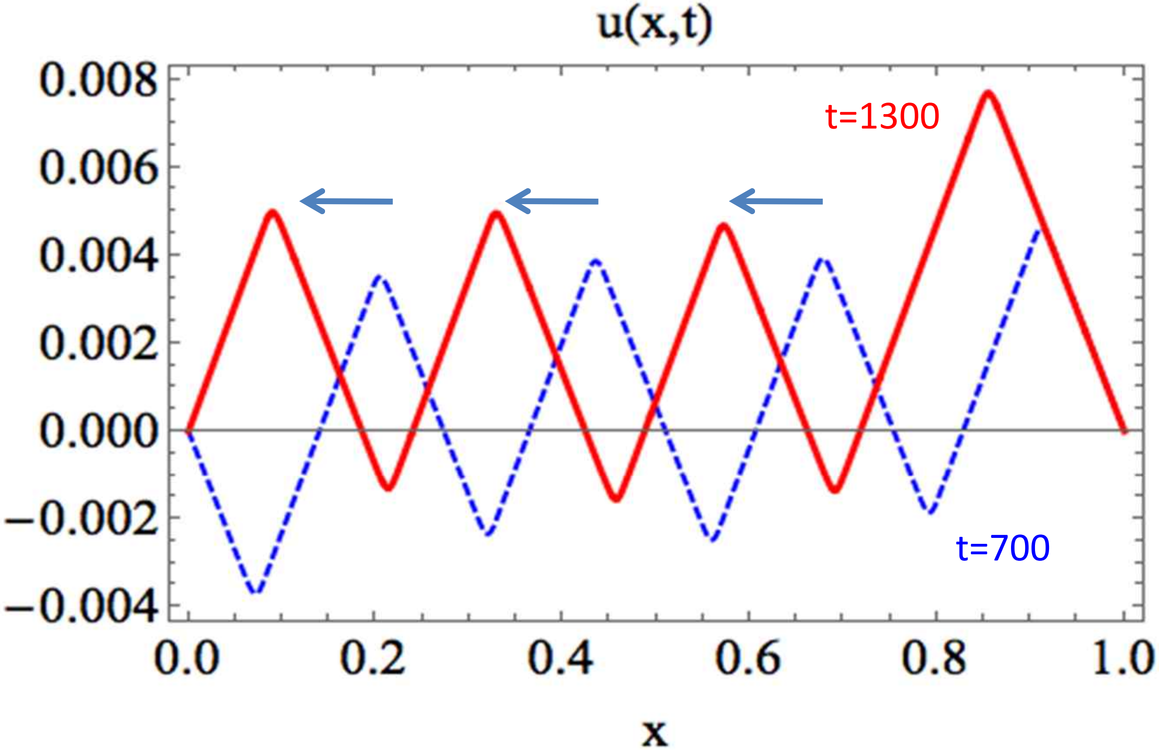




Density
+
Vorticity
lattices



Dynamic Staircases

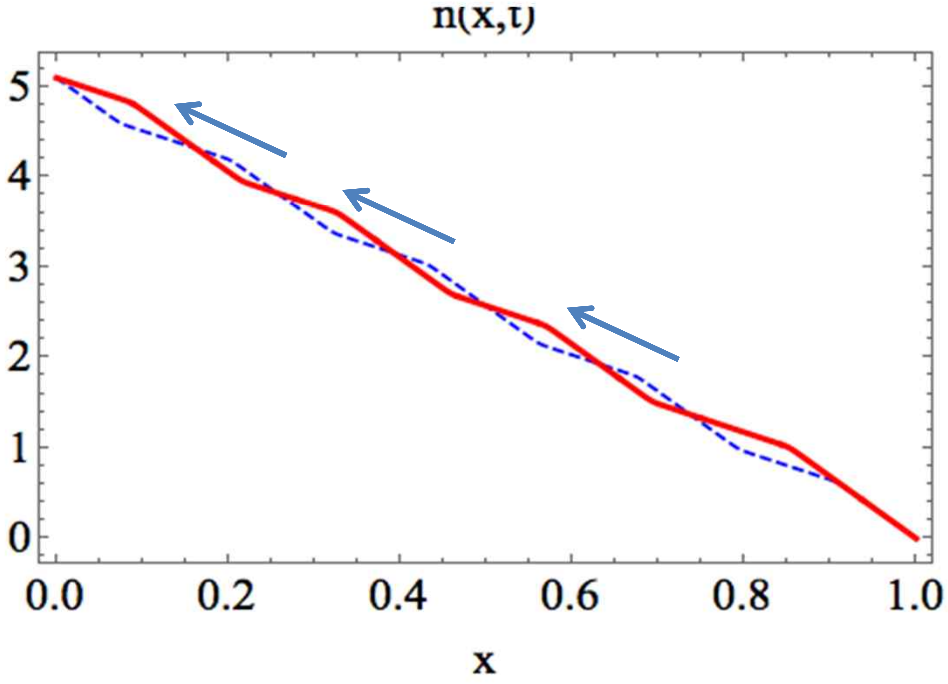


○Shear pattern detaches and delocalizes from its initial position of formation.

○Mesoscale shear lattice moves in the up-gradient direction. Shear layers condense and disappear at x=0.

○Shear lattice propagation takes place over much longer times. From t~O(10) to t~(10^4).

○Barriers in density profile move upward in an "<u>Escalator-like</u>" motion.



➔ Macroscopic Profile Re-structuring

↕

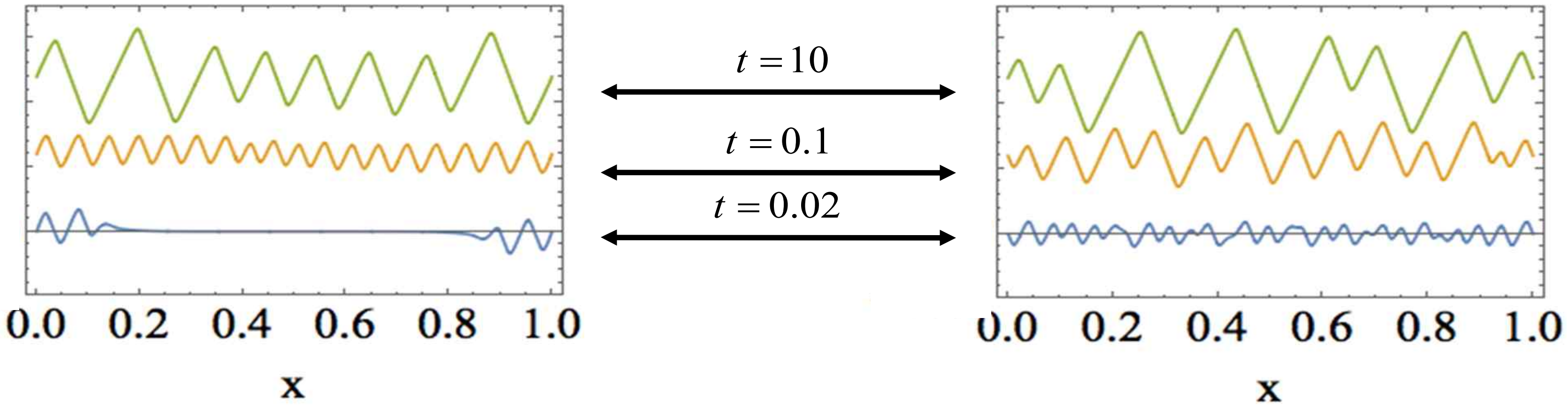
'Non-locality'

Mergers Occur

Nonlinear features develop from ‘linear’ instabilities

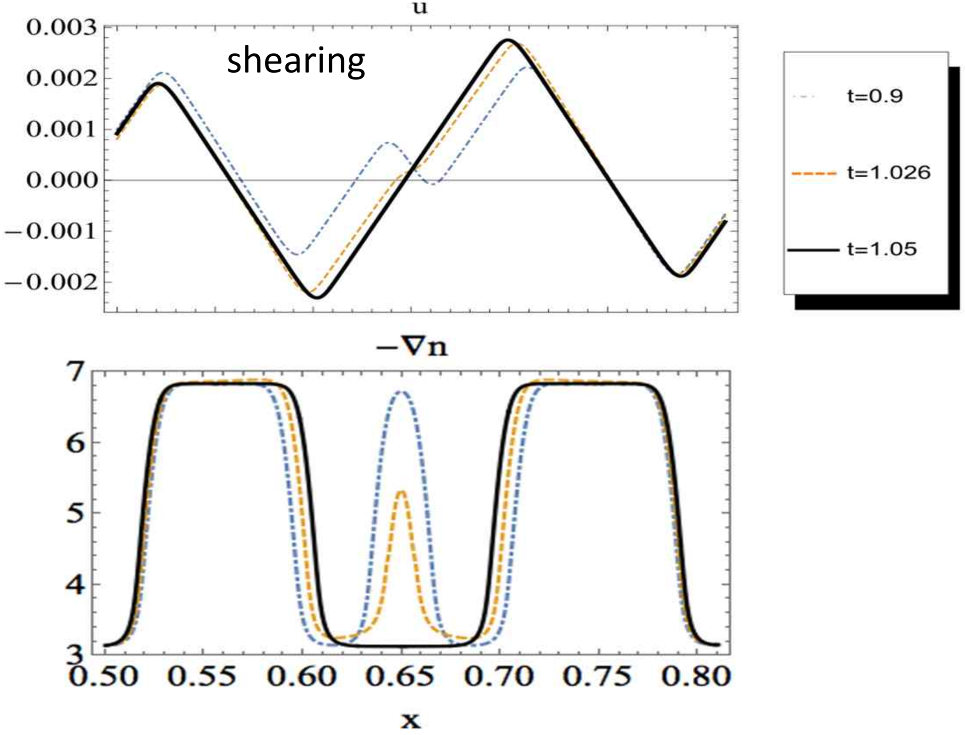
$\varepsilon(x=0,1)=0$

$\partial_x \varepsilon(x=0,1)=0$

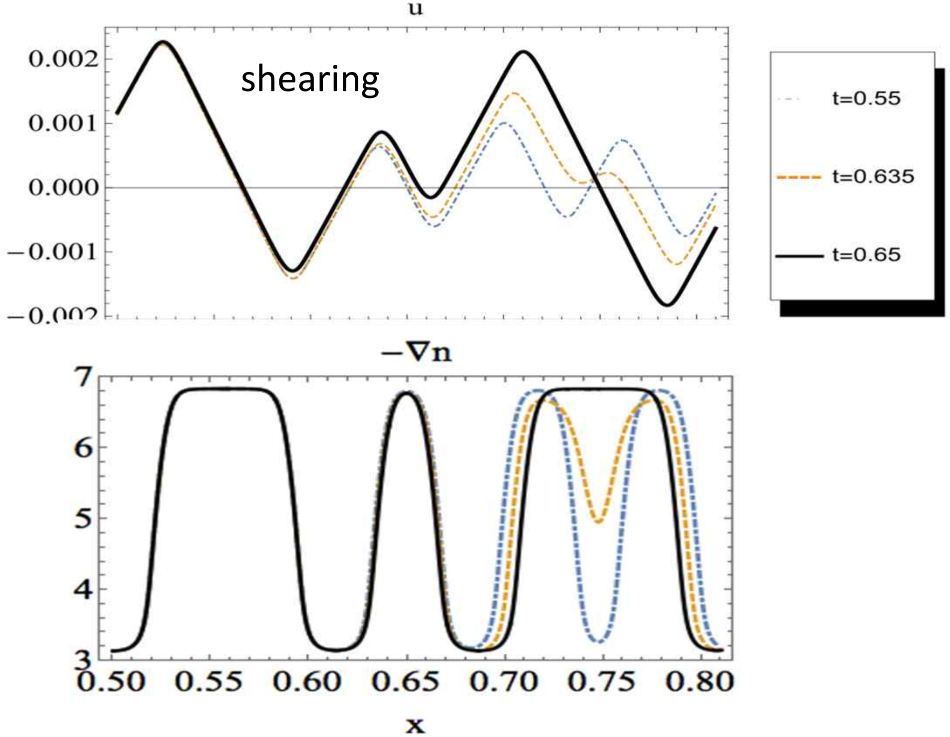


Local profile reorganization: Steps and jumps merge (continues up to times t~O(10))

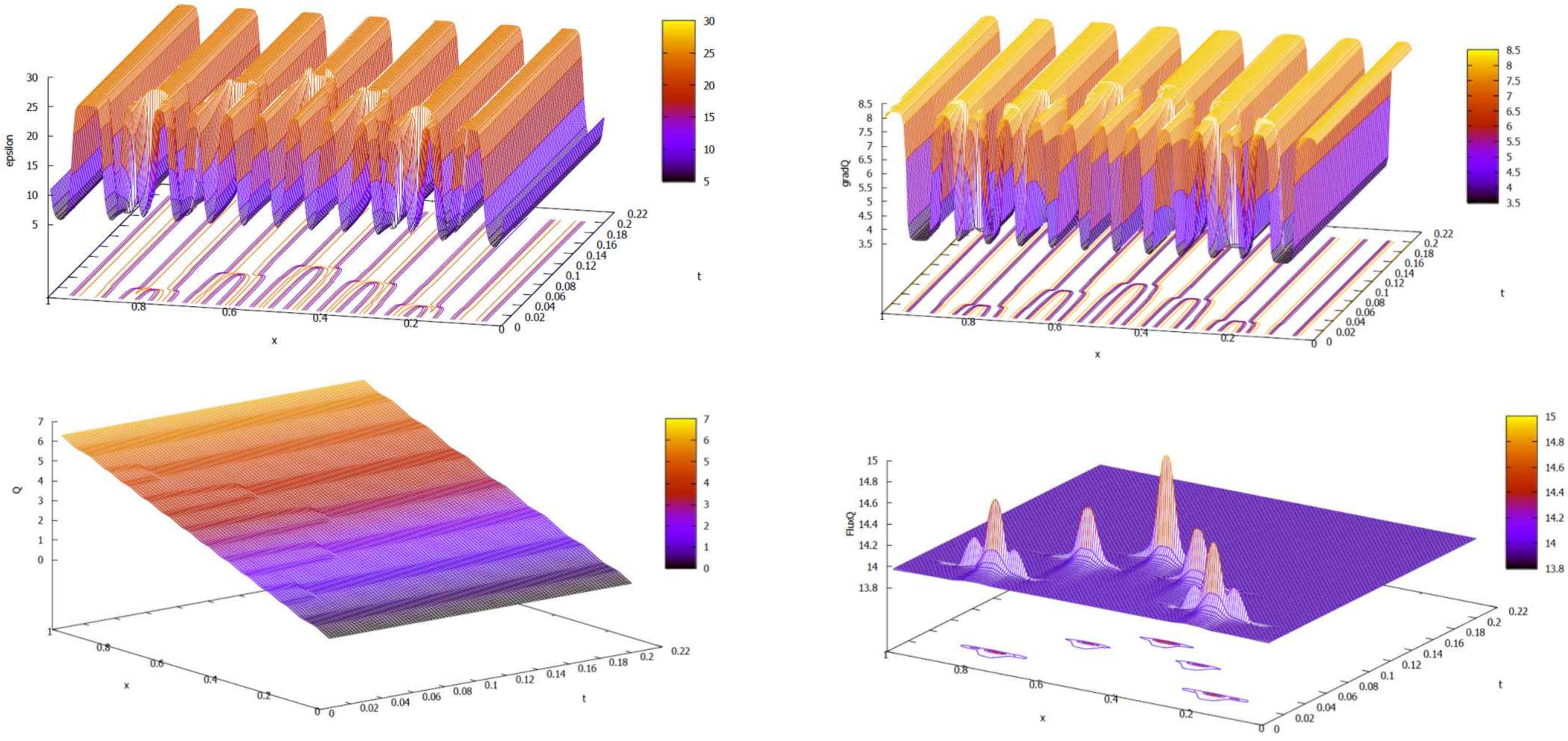
Merger between steps



Merger between jumps



Illustrating the merger sequence (QG-HM)



- ϵ
- Q_y

} top

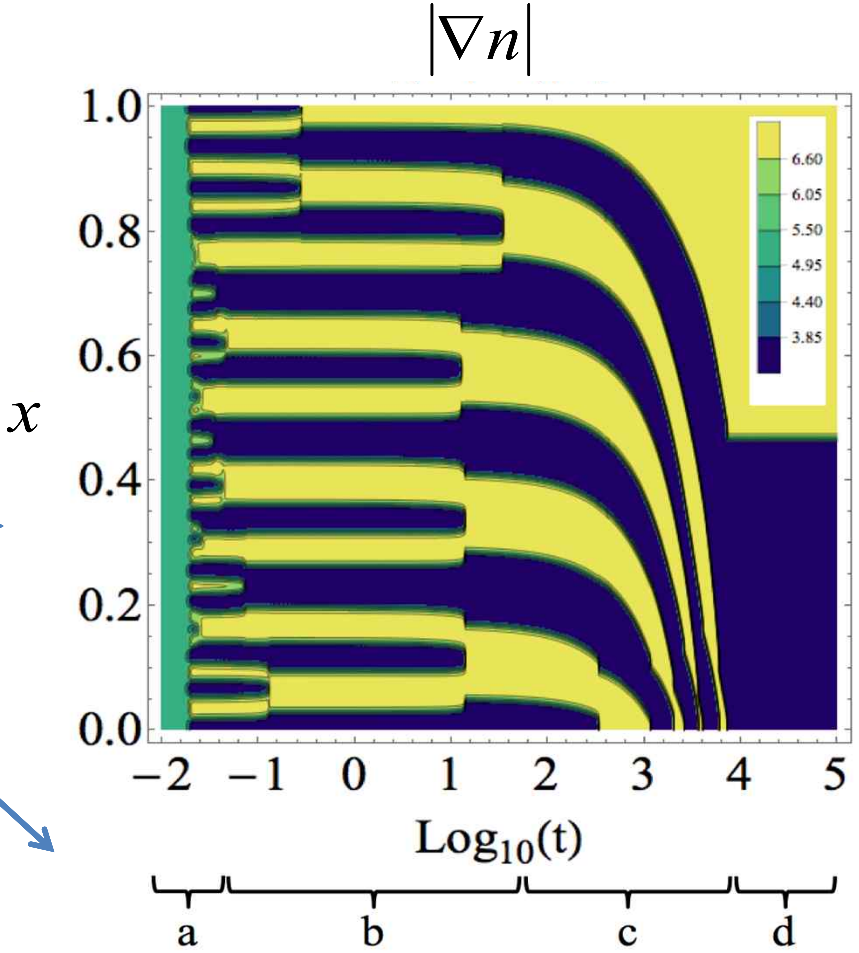
- Q
- Γ_q

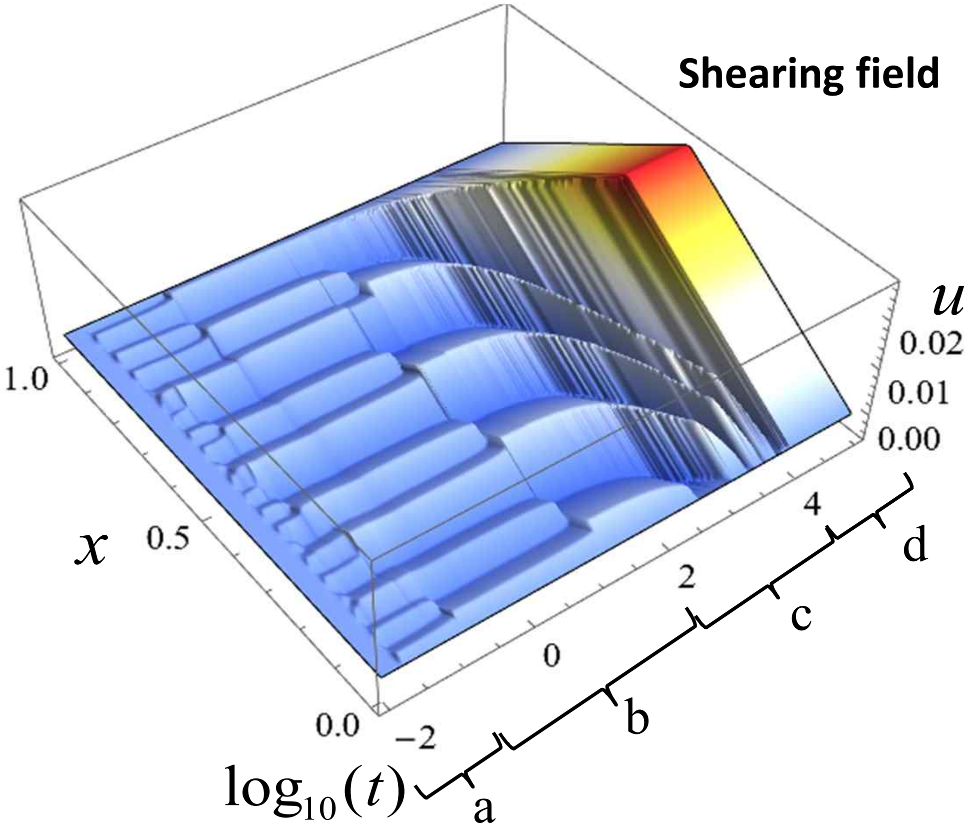
} bottom

Note later staircase mergers induce strong flux episodes!

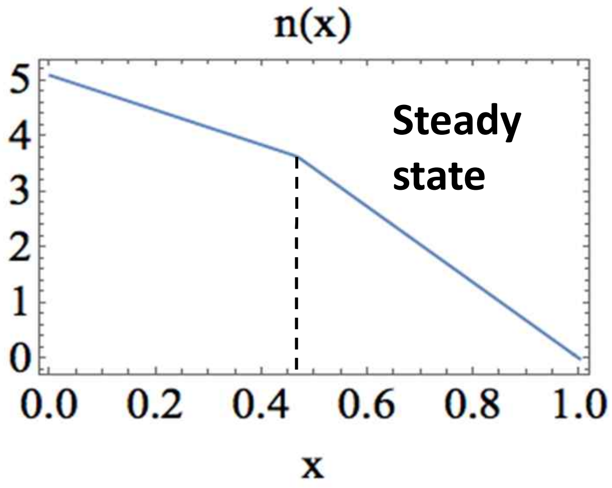
Time evolution of profiles

- (a) **Fast merger of micro-scale SC. Formation of meso-SC.**
- (b) **Meso-SC coalesce to barriers**
- (c) **Barriers propagate along gradient, condense at boundaries**
- (d) **Macro-scale stationary profile**





Shearing field



- The Point:
  - Macroscopic barrier emerges from hierarchical sequence of mergers and propagation, condensation
  - (Somewhat) familiar bi-stable transport model

  But

  - Barrier formation is NOT a local process

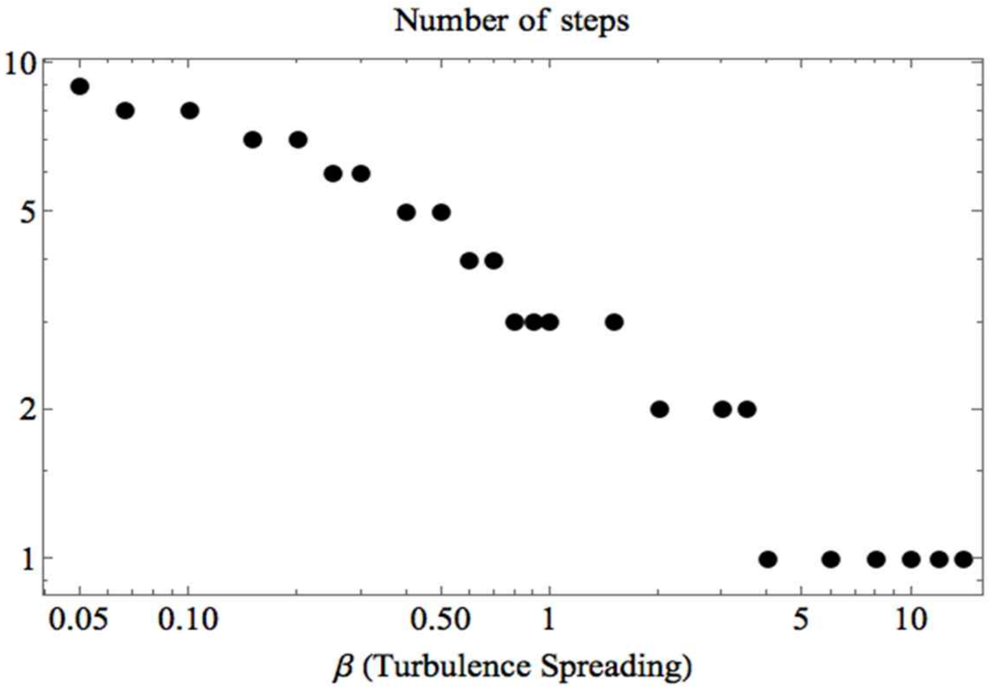
  → Begs for flux driven, but BVP analysis

Role of Turbulence Spreading

○ **Large turbulence spreading wipes out features on smaller spatial scales in the mean field profiles, resulting in the formation of fewer density and vorticity jumps.**

$$\partial_t \varepsilon = \beta \partial_x [(l^2 \varepsilon^{1/2}) \partial_x \varepsilon] + \ldots$$

- $\beta \to 0$ excessive profile roughness



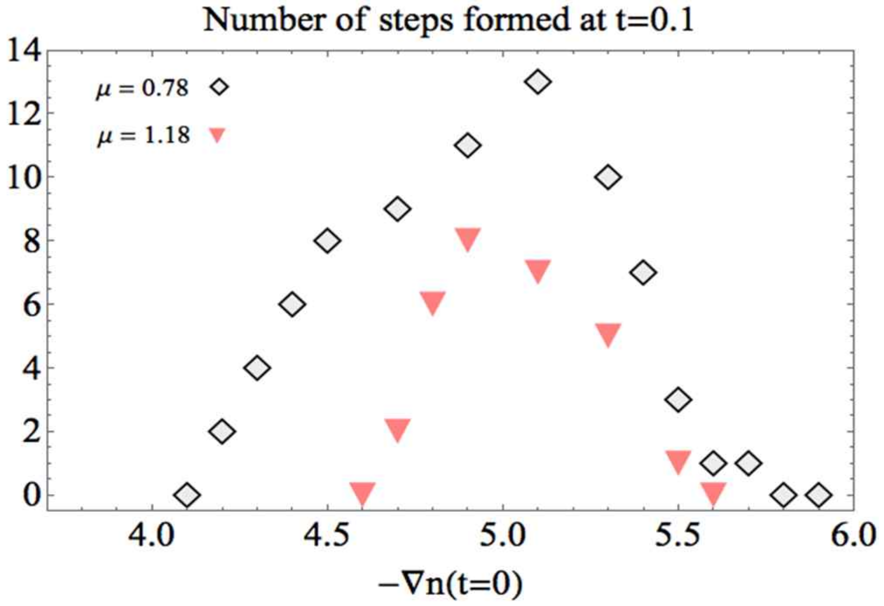
Initial condition dependence

○**Solutions are not sensitive to initial value of turbulent PE.**

○**Initial density gradient is the parameter influencing the subsequent evolution in the system.**

○**At lower viscosity more steps form.**

○**Width of density jumps grows with the initial density gradient.**



Lessons

- Staircases happen
  - Staircase is ‘natural upshot’ of modulation in bistable/multi-stable system
  - Bistability is a consequence of mixing scale dependence on gradients, intensity ←→ define feedback process
  - Bistability effectively locks in inhomogeneous PV mixing required for zonal flow formation
  - Mergers result from accommodation between boundary condition, drive(L), initial secondary instability
  - Staircase is natural extension of quasi-linear modulational instabilty/predator-prey model → couples to transport and b.c. ←→ simple natural phenomenon

QG – Vlasov Correspondence

- Resonance → granulation ?!
- Vlasov staircase from inhomogeneous mixing of f. Coarse graining a must
- Velocity space barrier?
- Beyond weakly nonlinear momentum theorem → Pseudomomentum for Vlasov system and its meaning