Memory, Cascades and Spectra in Models of 2D MHD and Elastic Turbulence

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Why Care?

- This is not a traditional study on transport in confined plasma. It is about the fundamental physics of memory and cascades in **elastic turbulence systems**. Why should we care?

- Virtually all models of drift-Alfven, EM ITG, etc. turbulence are based upon a vorticity equation, Ohm's Law and (usually multiple) scalar advection equations. The appearance of the Alfven wave introduces a crucial element of *memory* to the dynamics. Such Alfvenization-induced-memory can significantly impact structure formation and transport in turbulence.

- In this poster, we study **the fundamental physics of memory and cascades in very simple models** such as 2D MHD and elastic turbulence.

Why Care?

- The 2D Cahn-Hilliard Navier-Stokes (CHNS) model for spinodal decomposition is an example of generalized elastic system, and is analogous to 2D MHD.

- 2D MHD is simple but similar to some important models for transport studies.

2D CHNS
$$\begin{cases} \partial_t \psi + \vec{v} \cdot \nabla \psi = D\nabla^2(-\psi + \psi^3 - \xi^2 \nabla^2 \psi) \\ \partial_t \omega + \vec{v} \cdot \nabla \omega = \dfrac{\xi^2}{\rho} \vec{B}_\psi \cdot \nabla \nabla^2 \psi + \nu \nabla^2 \omega \end{cases}$$

2D MHD
$$\begin{cases} \partial_t A + \vec{v} \cdot \nabla A = \eta \nabla^2 A \\ \partial_t \omega + \vec{v} \cdot \nabla \omega = \dfrac{1}{\mu_0 \rho} \vec{B} \cdot \nabla \nabla^2 A + \nu \nabla^2 \omega \end{cases}$$
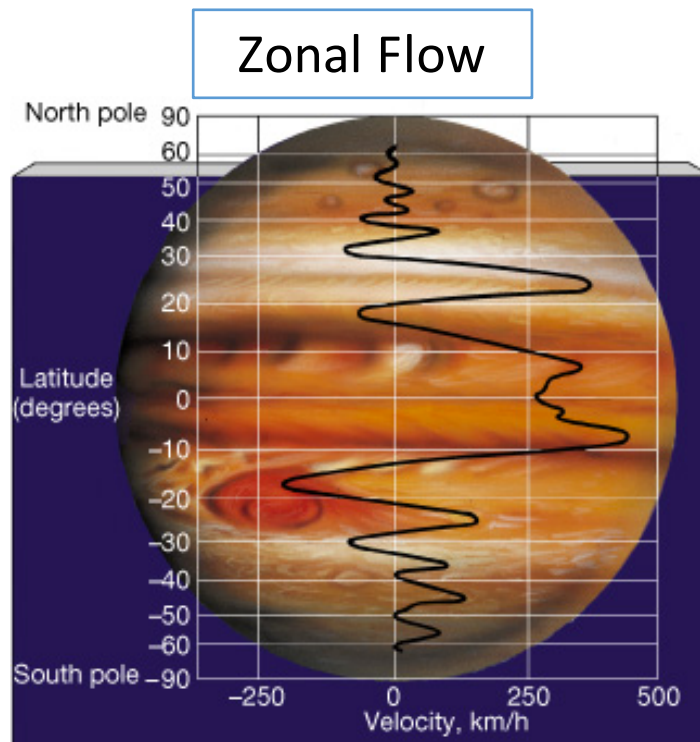
A typical 3-field model for ELM [Xi 2012]
$$\begin{cases} \dfrac{\partial A_\parallel}{\partial t} = -\partial_\parallel \Phi + \dfrac{\eta}{\mu_0} \nabla_\perp^2 A_\parallel - \dfrac{\eta_H}{\mu_0} \nabla_\perp^4 A_\parallel, \\ \dfrac{d\varpi}{dt} + \mathbf{V}_1 \cdot \nabla \varpi_0 = B_0 \nabla_\parallel J_\parallel + 2\mathbf{b}_0 \times \kappa \cdot \nabla P, \\ \dfrac{dP}{dt} + \mathbf{V}_1 \cdot \nabla P_0 = 0, \end{cases}$$
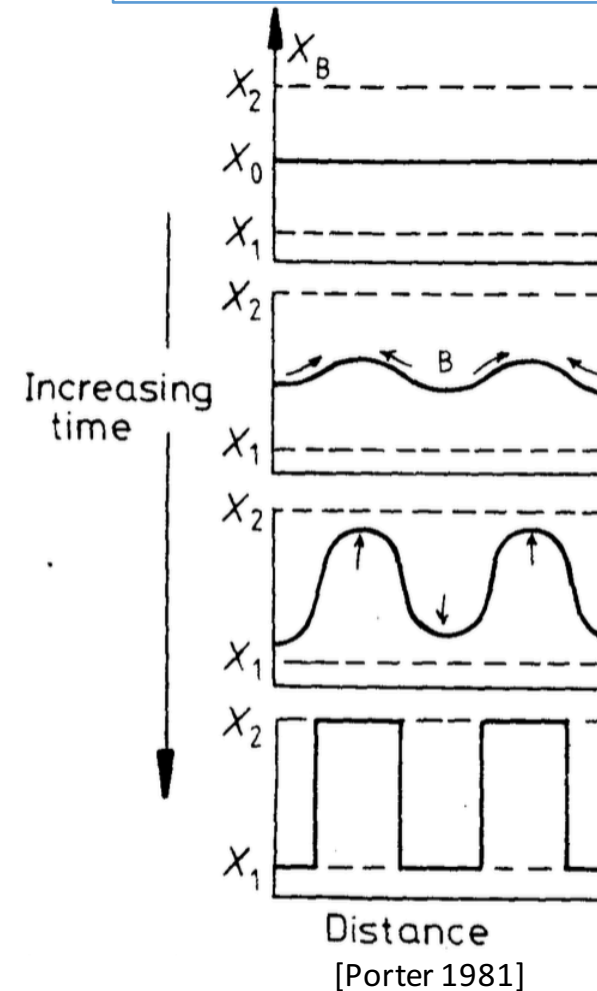
Why Care?

- Spinodal decomposition is also related to zonal flow.

- The pattern formation is because of negative diffusion/negative viscosity.

Spinodal Decomposition

[Porter 1981]

Zonal Flow

http://astronomy.nju.edu.cn/~lixd/GA/AT4/AT411/HTML/AT41102.htm

Overview

- The Cahn-Hilliard Navier-Stokes (CHNS) model for spinodal decomposition in 2D symmetric binary liquid mixture and 2D Magnetohydrodynamics (MHD) are both examples of generalized elastic turbulent systems.

- Since 2D MHD turbulence has been well studied, it provides us with potential insight and guidance for exploring the physics of 2D CHNS turbulence.

- We compare and contrast the _cascades and spectra_ of the two systems.

- **[New]** By direct numerical simulation, we find that in the elastic range, the mean square concentration spectrum H_k^{ψ} of the 2D CHNS system exhibits the same power law (−7/3) as the mean square magnetic potential spectrum H_k^A in the inverse cascade regime of 2D MHD.

- **[New]** The kinetic energy spectrum of the 2D CHNS system is $E_k^K \sim k^{-3}$ if forced at large scale, suggestive of the direct enstrophy cascade power law of 2D Navier-Stokes (NS) turbulence. It is different from 2D MHD, and the difference could be explained by the difference of interface packing fraction.

Outline

- Introduction: What is Spinodal Decomposition
- The CHNS model for spinodal decomposition
- 2D CHNS turbulence compared to 2D MHD turbulence
  - Basic equations
  - Ideal quadratic conserved quantities
  - Cascades
  - Linear elastic wave
- Important length scales and ranges of 2D CHNS turbulence
- Simulation Results
  - Simulation Setup
  - Benchmark
  - H_k^ψ / H_k^A spectrum power law
  - Energy spectrum power law
- Conclusion & discussion

Introduction: What Is Spinodal Decomposition

- **What is spinodal decomposition**: binary liquid mixtures can pass spontaneously from one miscible phase to two coexisting immiscible phases following a temperature drop, and this second order phase transition is called the spinodal decomposition.

- One miscible phase: like water + alcohol.

- Two coexisting immiscible phases: like water + oil.

- A real life example: the cyclohexane methanol system, with the critical temperature of spinodal decomposition $T_C = 45.14°C$. [Huang et al. 1974]

- The Cahn-Hilliard Navier-Stokes (CHNS) model is the standard model for binary liquid mixture undergoing spinodal decomposition.

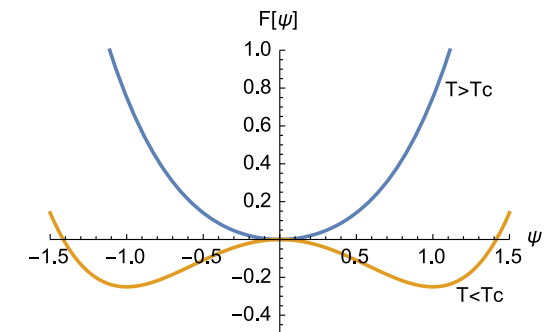# Introduction: What Is Spinodal Decomposition
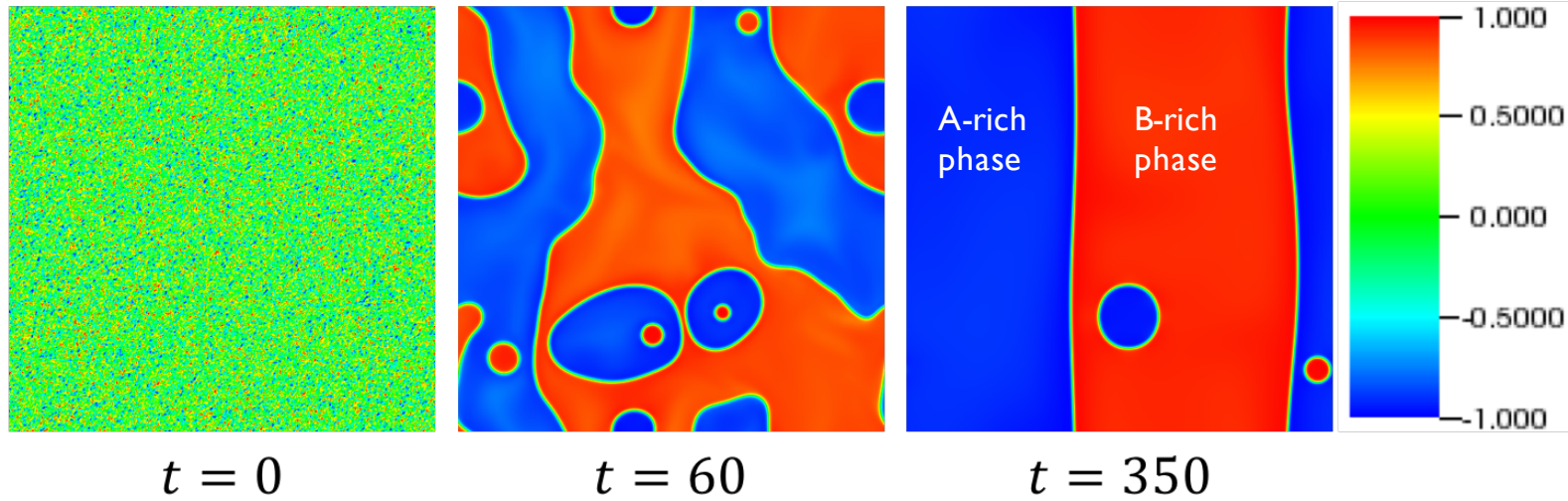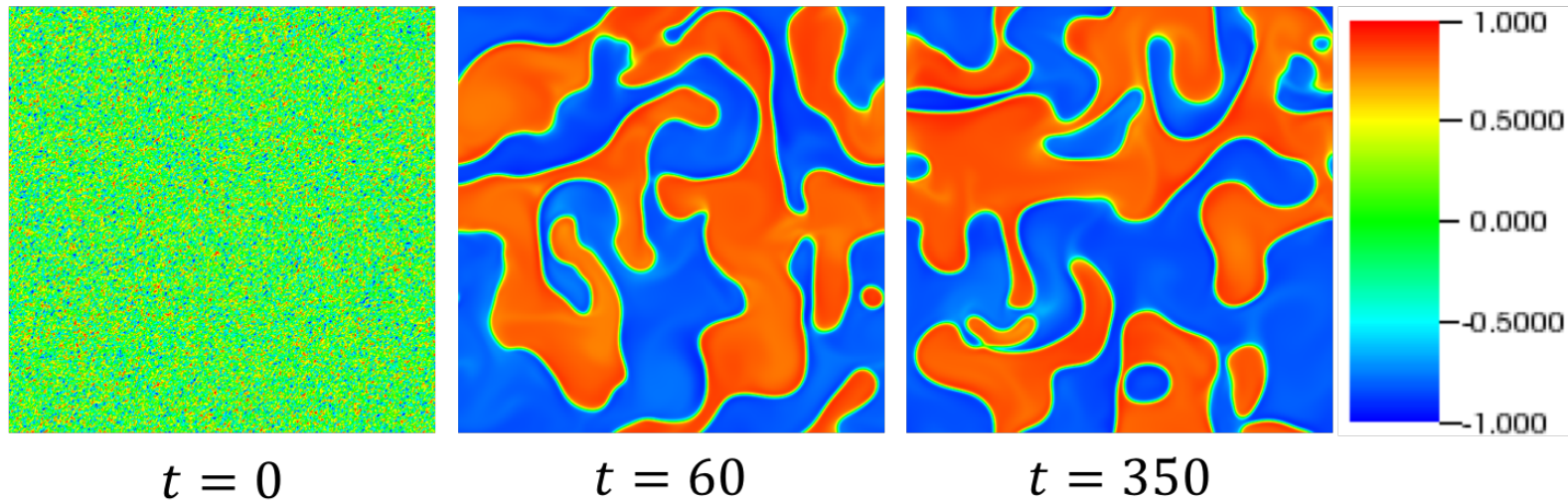
The CHNS Model for Spinodal Decomposition

- Second order phase transition -> Landau theory.

- The order parameter: the local relative concentration field:

$$\psi(\vec{r}, t) \overset{\text{def}}{=} [\rho_A(\vec{r}, t) - \rho_B(\vec{r}, t)]/\rho$$

- $\psi = -1$ means A-rich phase, $\psi = 1$ means B-rich phase.

- The free energy functional:

$$\mathrm{F}(\psi) = \int d\vec{r} \left(\frac{1}{2} A\psi^2 + \frac{1}{4} B\psi^4 + \frac{\xi^2}{2} |\nabla\psi|^2 \right)$$

$\underbrace{}_{\text{Phase Transition}}$ $\underbrace{}_{\text{Curvature Penalty}}$

Where A and B are coefficients of Taylor expansion, $A = A_0(T - T_c)$; ξ is a parameter that describes the strength of the interaction.

- For simplicity, we study the isothermal case when the temperature is fixed below T_c, and set B = −A = 1:

$$\mathrm{F}(\psi) = \int d\vec{r} \left(-\frac{1}{2}\psi^2 + \frac{1}{4}\psi^4 + \frac{\xi^2}{2} |\nabla\psi|^2 \right)$$

The CHNS Model for Spinodal Decomposition

- Chemical potential: $\mu = \frac{\delta F(\psi)}{\delta \psi} = -\psi + \psi^3 - \xi^2 \nabla^2 \psi$. Fick's Law: $\vec{J} = -D\nabla\mu$.

  Continuity equation: $\frac{d\psi}{dt} + \nabla \cdot \vec{J} = 0$. Combining the above together:

$$\frac{d\psi}{dt} = D\nabla^2\mu = D\nabla^2(-\psi + \psi^3 - \xi^2\nabla^2\psi) \quad \boxed{\text{Cahn-Hilliard Equation}}$$

- The fluid velocity comes in via the convection term $d_t = \partial_t + \vec{v} \cdot \nabla$.
- The surface tension enters the Navier-Stokes Eq as a force:

$$\partial_t\vec{v} + \vec{v} \cdot \nabla\vec{v} = -\frac{\nabla p}{\rho} - \psi\nabla\mu + \nu\nabla^2\vec{v}$$

- For incompressible fluid, it is convenient to take the curl and work with vorticity. Finally, we get the Cahn-Hilliard Navier-Stokes (CNHS) Equations:

$$\partial_t\psi + \vec{v} \cdot \nabla\psi = D\nabla^2(-\psi + \psi^3 - \xi^2\nabla^2\psi)$$

$$\partial_t\omega + \vec{v} \cdot \nabla\omega = \frac{\xi^2}{\rho}\vec{B}_\psi \cdot \nabla\nabla^2\psi + \nu\nabla^2\omega$$

With $\vec{v} = \hat{\vec{z}} \times \nabla\phi$, $\omega = \nabla^2\phi$, $\vec{B}_\psi = \hat{\vec{z}} \times \nabla\psi$, $j_\psi = \xi^2\nabla^2\psi$.

Comparison & Contrast: Basic Equations

- The 2D CHNS Equations:

$$\partial_t \psi + \vec{v} \cdot \nabla \psi = D\nabla^2(-\psi + \psi^3 - \xi^2\nabla^2\psi)$$

$$\partial_t \omega + \vec{v} \cdot \nabla \omega = \frac{\xi^2}{\rho}\vec{B}_\psi \cdot \nabla\nabla^2\psi + \nu\nabla^2\omega$$

$-\psi$: Negative diffusion term
ψ^3: Self nonlinear term
$-\xi^2\nabla^2\psi$: Hyper-diffusion term

With $\vec{v} = \hat{\vec{z}} \times \nabla\phi$, $\omega = \nabla^2\phi$, $\vec{B}_\psi = \hat{\vec{z}} \times \nabla\psi$, $j_\psi = \xi^2\nabla^2\psi$

- The 2D MHD Equations:

$$\partial_t A + \vec{v} \cdot \nabla A = \eta\nabla^2 A$$

$$\partial_t \omega + \vec{v} \cdot \nabla \omega = \frac{1}{\mu_0\rho}\vec{B} \cdot \nabla\nabla^2 A + \nu\nabla^2\omega$$

A: Simple diffusion term

With $\vec{v} = \hat{\vec{z}} \times \nabla\phi$, $\omega = \nabla^2\phi$, $\vec{B} = \hat{\vec{z}} \times \nabla A$, $j = \frac{1}{\mu_0}\nabla^2 A$

- The force on fluid: $\dfrac{\xi^2}{\rho}\vec{B}_\psi \cdot \nabla\nabla^2\psi \Longleftrightarrow \dfrac{1}{\mu_0\rho}\vec{B} \cdot \nabla\nabla^2 A$

- Note that the magnetic potential A is a scalar in 2D.

	2D MHD	2D CHNS
Magnetic Potential	A	ψ
Magnetic Field	$\mathbf{B}$	$\mathbf{B}_\psi$
Current	j	j_ψ
Diffusivity	η	D
Interaction strength	$\frac{1}{\mu_0}$	ξ^2

Comparison & Contrast:
Ideal quadratic conserved quantities

- **2D MHD**

1. Energy

$$E = E^K + E^B = \int \left(\frac{v^2}{2} + \frac{B^2}{2\mu_0}\right) d^2x$$

2. Mean Square Magnetic Potential

$$H^A = \int A^2 \, d^2x$$

3. Cross Helicity

$$H^C = \int \vec{v} \cdot \vec{B} \, d^2x$$

- **2D CHNS**

1. Energy

$$E = E^K + E^B = \int \left(\frac{v^2}{2} + \frac{\xi^2 B_\psi^2}{2}\right) d^2x$$

2. Mean Square Concentration

$$H^\psi = \int \psi^2 \, d^2x$$

3. Cross Helicity

$$H^C = \int \vec{v} \cdot \vec{B}_\psi \, d^2x$$

- "Ideal" here means $D, \eta = 0; \nu = 0$.

Comparison & Contrast: Cascades

- Turbulence cascade directions are suggested by the absolute equilibrium distributions.

- The peak of the absolute equilibrium distribution for each quadratic conserved quantity is a good indicator of the corresponding cascade direction.

- The spectrum is peaked at high k -> excitation relaxes towards high k -> direct cascade.

- The spectrum is peaked at small k -> excitation relaxes towards small k -> inverse cascade.

- This approach only depends on the ideal quadratic conserved quantities of the system, so we can follow a similar approach as for 2D MHD to obtain an indication of the cascade directions in 2D CHNS.

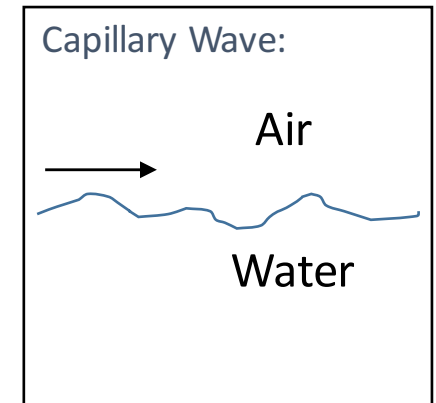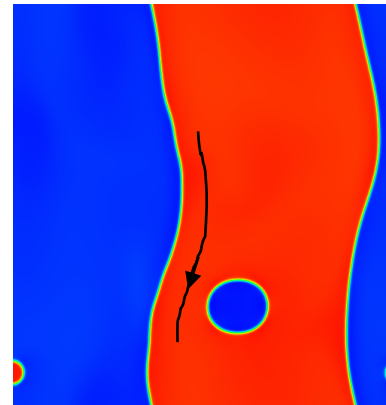| Physics System	Conserved Quantity	Cascade Direction
2D MHD	E_k	Direct
	H_k^A	Inverse
2D CHNS	E_k	Direct
	H_k^ψ	Inverse
2D NS	E_k^K	Inverse
	Ω_k	Direct

Comparison & Contrast: Linear Elastic Wave

- Alfven wave in 2D MHD:

$$\omega(k) = \pm\sqrt{\frac{1}{\mu_0\rho}\left|\vec{k}\times\vec{B}_0\right|} - \frac{1}{2}i(\eta + \nu)k^2$$

- Linear elastic wave in 2D CHNS:

$$\omega(k) = \pm\sqrt{\frac{\xi^2}{\rho}\left|\vec{k}\times\vec{B}_{\psi 0}\right|} - \frac{1}{2}i(CD + \nu)k^2$$

Capillary Wave:

Air

Water

Where $C \equiv [-1 - 6\psi_0\nabla^2\psi_0/k^2 - 6(\nabla\psi_0)^2/k^2 - 6\psi_0\nabla\psi_0\cdot i\mathbf{k}/k^2 + 3\psi_0^2 + \xi^2 k^2]$

- The linear elastic wave in 2D CHNS is like a capillary wave: it only propagates along the boundary of the two fluids, where the gradient of concentration $\vec{B}_{\psi 0} \neq 0$. Surface tension generates restoring force.

- The wave is similar to Alfven wave: they have similar dispersion relation; they both propagates along $\vec{B}_0$ field lines; both magnetic field and surface tension act like an elastic restoring force.

- Important difference: B fills the whole space; $\vec{B}_{\psi 0}$ is large only in the interface regions.

Important length scales and ranges

- **Hinze scale**: the balance between turbulent kinetic energy and surface tension energy: $\frac{\rho \langle v^2 \rangle}{\sigma / L_H} \sim 1$ where σ is surface tension and $\sigma = \sqrt{8/9}\xi$ [Kendon et al. 2001].

- In the 2D NS direct enstrophy cascade regime, the velocity distribution is $\frac{\langle v^2 \rangle}{k_H} \sim \epsilon_\Omega^{2/3} k_H^{-3}$. So

$$L_H \sim (\frac{\rho}{\xi})^{-1/3} \epsilon_\Omega^{-2/9}$$

- We define the scales between L_H and dissipation scale L_d to be **the elastic range**, where the blob coalescence process dominates.

- Define a dimensionless number: $\frac{L_H}{L_d} = Hd \sim (\frac{\rho}{\xi})^{-1/3} v^{-1/2} \epsilon_\Omega^{-1/18}$

- $\frac{L_H}{L_d} = Hd \gg 1$ is required to form a long enough elastic range.

Important length scales and ranges
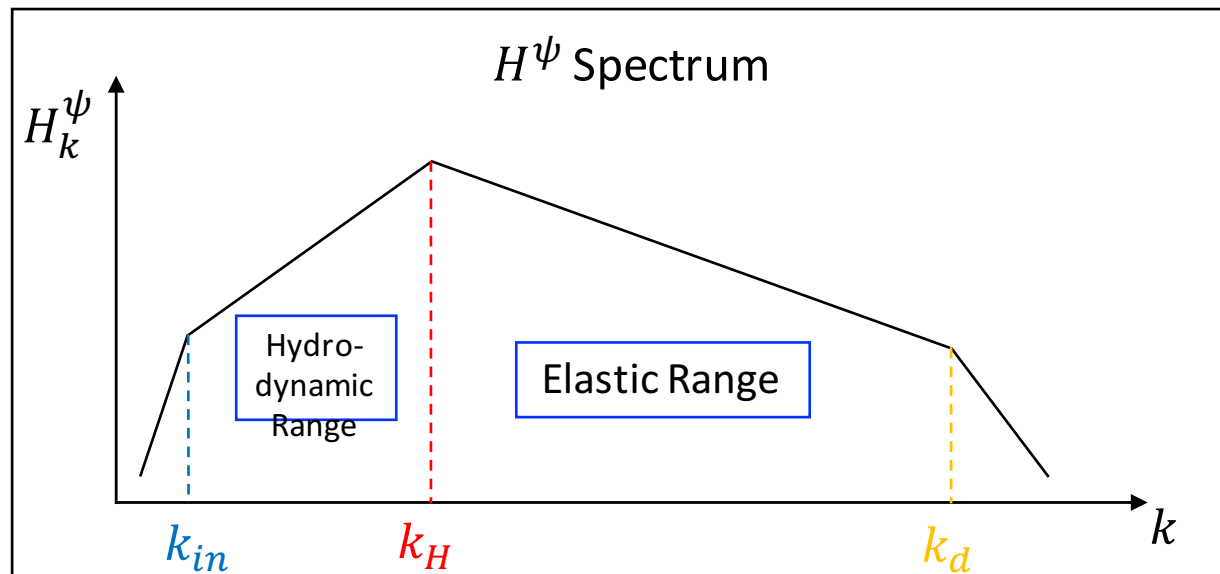
- In the elastic range of the 2D CHNS system, the blob coalescence process is analogous to the magnetic flux blob coalescence process in 2D MHD.

- The former leads to the inverse cascade of H^ψ, and the latter leads to the inverse cascade of H^A.

- In the elastic range of the 2D CHNS system, surface tension induces elasticity and plays a major role in defining a restoring force. Similarly, in 2D MHD, the magnetic field induces elasticity and make MHD different from a pure fluid.

- The 2D CHNS system is more MHD-like in the elastic range.

Simulation Setup

- The PIXIE2D code [Chacon 2002, 2003] is used to simulate the system. PIXIE2D originally solves the 2D MHD equation, and now is modified to be able to solve the Cahn-Hilliard Navier-Stokes (CHNS) equations, too. It is a Direct Numerical Simulation (DNS) code that solves the following equations in real space:

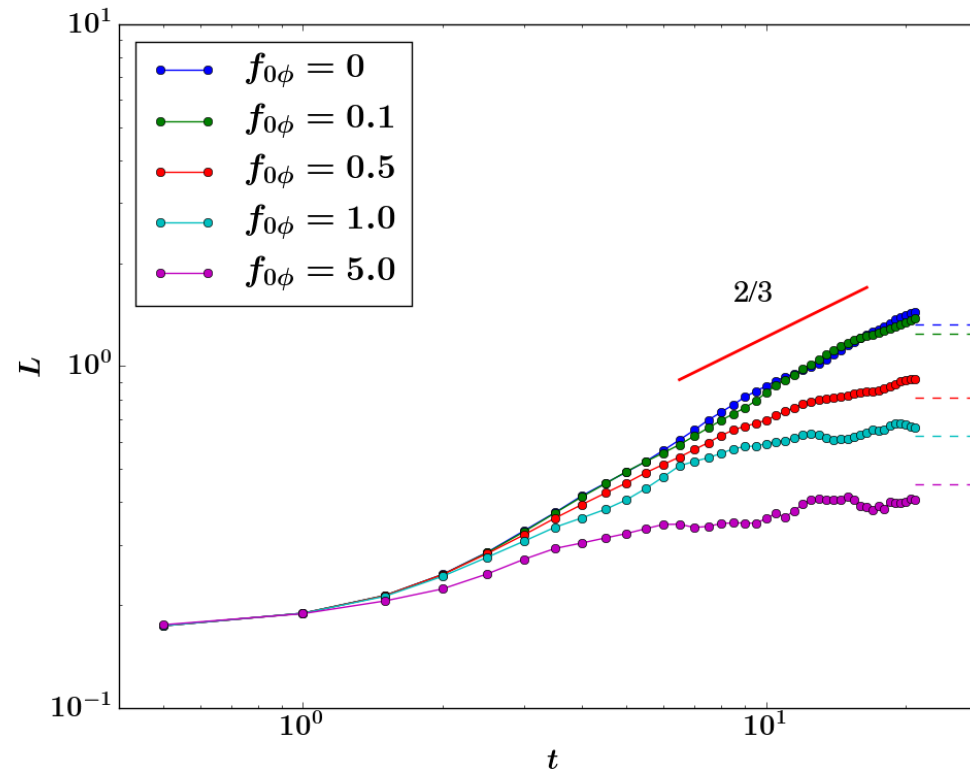$$\partial_t \psi + \vec{v} \cdot \nabla \psi = D \nabla^2 (-\psi + \psi^3 - \xi^2 \nabla^2 \psi) + F_\psi$$

$$\partial_t \omega + \vec{v} \cdot \nabla \omega = \frac{\xi^2}{\rho} \vec{B}_\psi \cdot \nabla \nabla^2 \psi + \nu \nabla^2 \omega + F_\omega$$

With $\vec{v} = \hat{\vec{z}} \times \nabla \phi$, $\omega = \nabla^2 \phi$, $\vec{B}_\psi = \hat{\vec{z}} \times \nabla \psi$, $j_\psi = \xi^2 \nabla^2 \psi$

- Initial condition: ψ in each cell is assigned to 1 or -1 randomly; $\phi = 0$ everywhere.

- Boundary condition: doubly periodic.

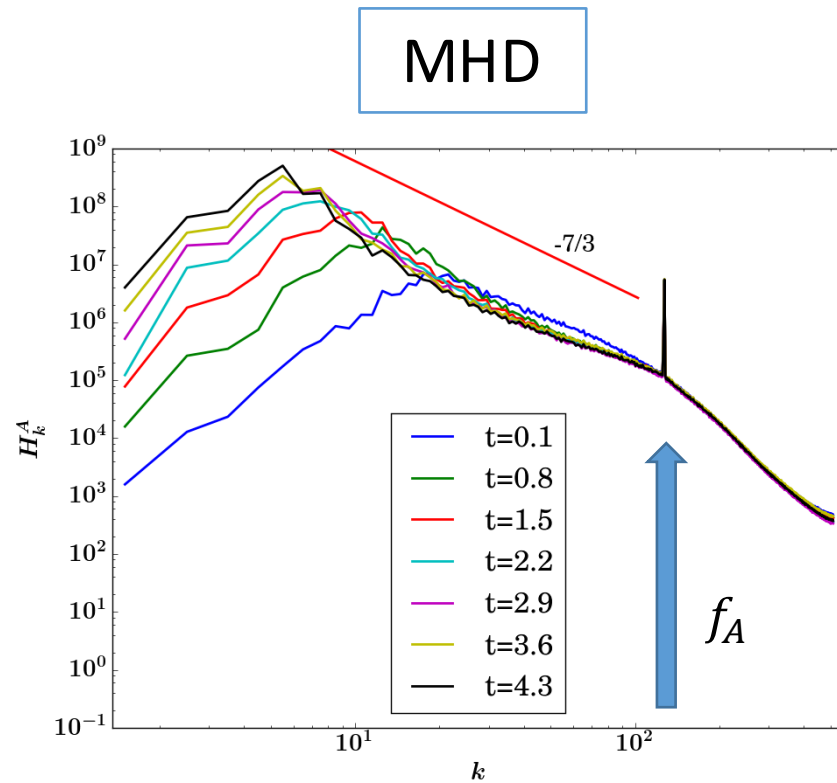- External force for either ψ or ϕ: an isotropic homogeneous force that has a wave number k_{in}:

$$f_{A,\psi,\phi}(x,y) = f_{0A,\psi,\phi} \sin[x * \text{int}(k_{fA,\psi,\phi} \cos \theta_{A,\psi,\phi}) + y * \text{int}(k_{fA,\psi,\phi} \sin \theta_{A,\psi,\phi}) + \varphi_{A,\psi,\phi}]$$
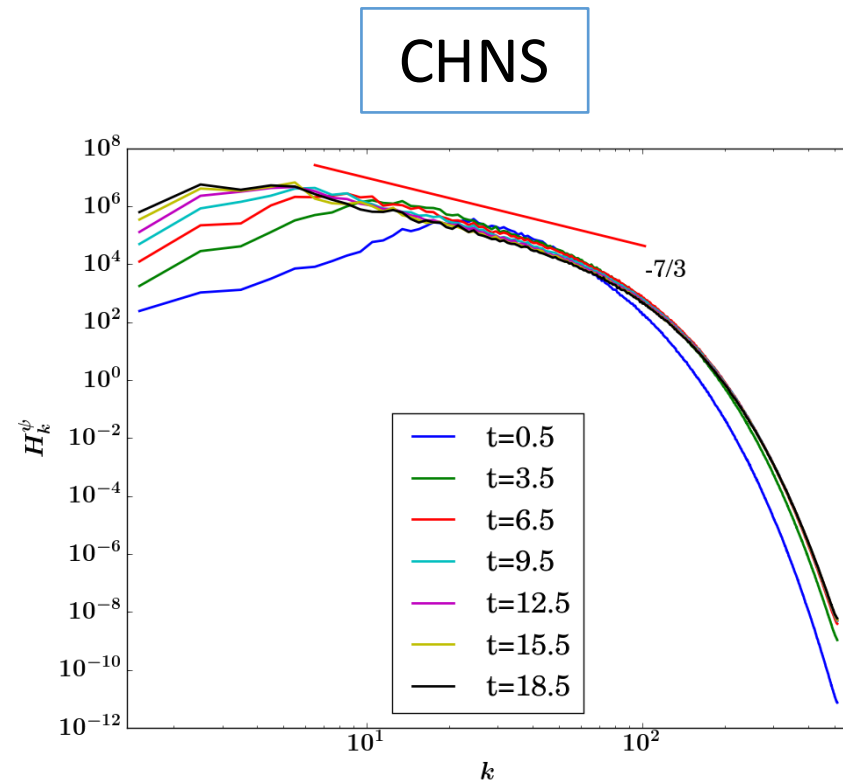
Benchmark

- We verified the length scale growth is $L(t) \sim t^{2/3}$; and the growth can be arrested. The saturation length scale is consistent with the Hinze scale (dashed line).

H_k^ψ / H_k^A spectrum power law: -7/3

MHD

CHNS

$$H_k^A \sim \epsilon_{HA}^{2/3} k^{-7/3}$$

$$H_k^\psi \sim \epsilon_{H\psi}^{2/3} k^{-7/3}$$

Inverse cascade of H^A

Inverse cascade of H^ψ

H_k^ψ / H_k^A spectrum power law: -7/3

- Assuming a constant mean square magnetic potential dissipation rate ϵ_{HA}, according to the Alfvenic equipartition $(\rho\langle v^2\rangle \sim \frac{1}{\mu_0}\langle B^2\rangle)$, the time scale for the decay of H^A $(\epsilon_{HA} \sim H^A/\tau)$ can be estimated by $\tau \sim (vk)^{-1} \sim (Bk)^{-1}$.

- Define the spectrum to be $H^A = \sum_k H_k^A \sim k H_k^A$, so $B \sim kA \sim k(H^A)^{\frac{1}{2}} \sim (H_k^A)^{\frac{1}{2}} k^{\frac{3}{2}}$. Therefore $\epsilon_{HA} \sim \frac{H^A}{\tau} \sim (H_k^A)^{\frac{2}{3}} k^{\frac{7}{2}}$.
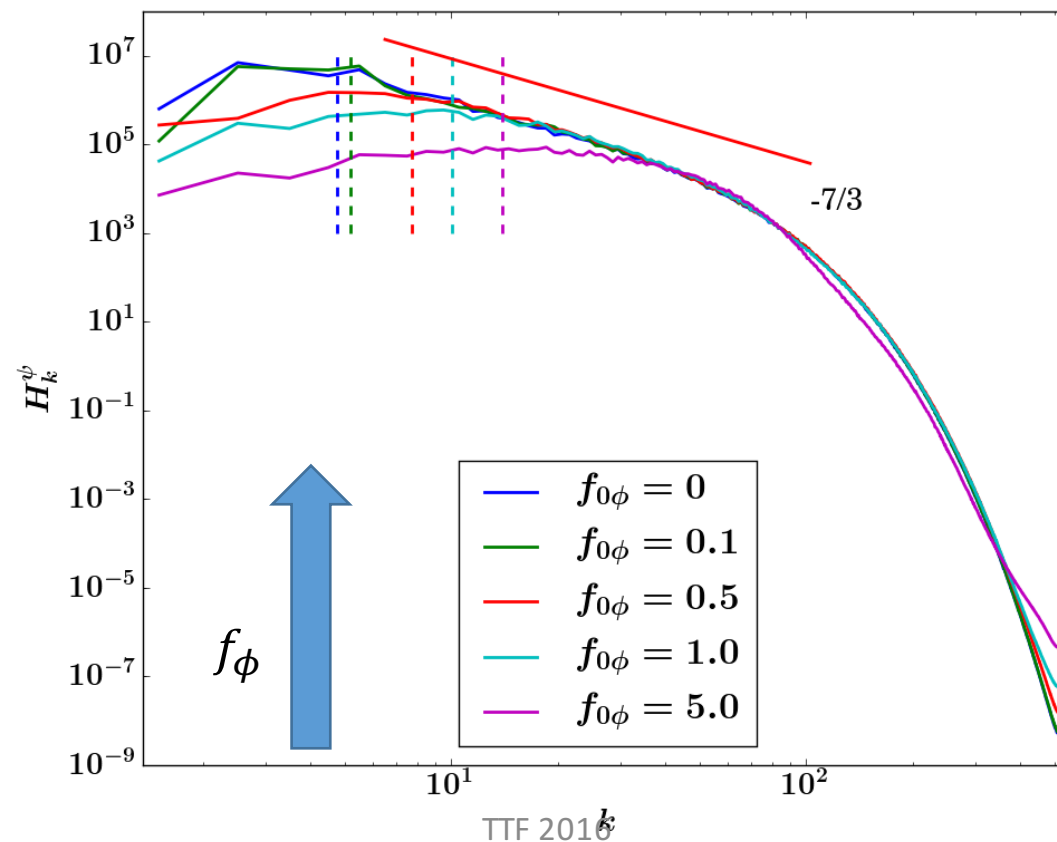
$$H_k^A \sim \epsilon_{HA}^{2/3} k^{-7/3}$$

- Similarly, we can obtain the H_k^ψ spectrum by assuming the elastic equipartition $(\rho\langle v^2\rangle \sim \xi^2\langle B_\psi^2\rangle)$:
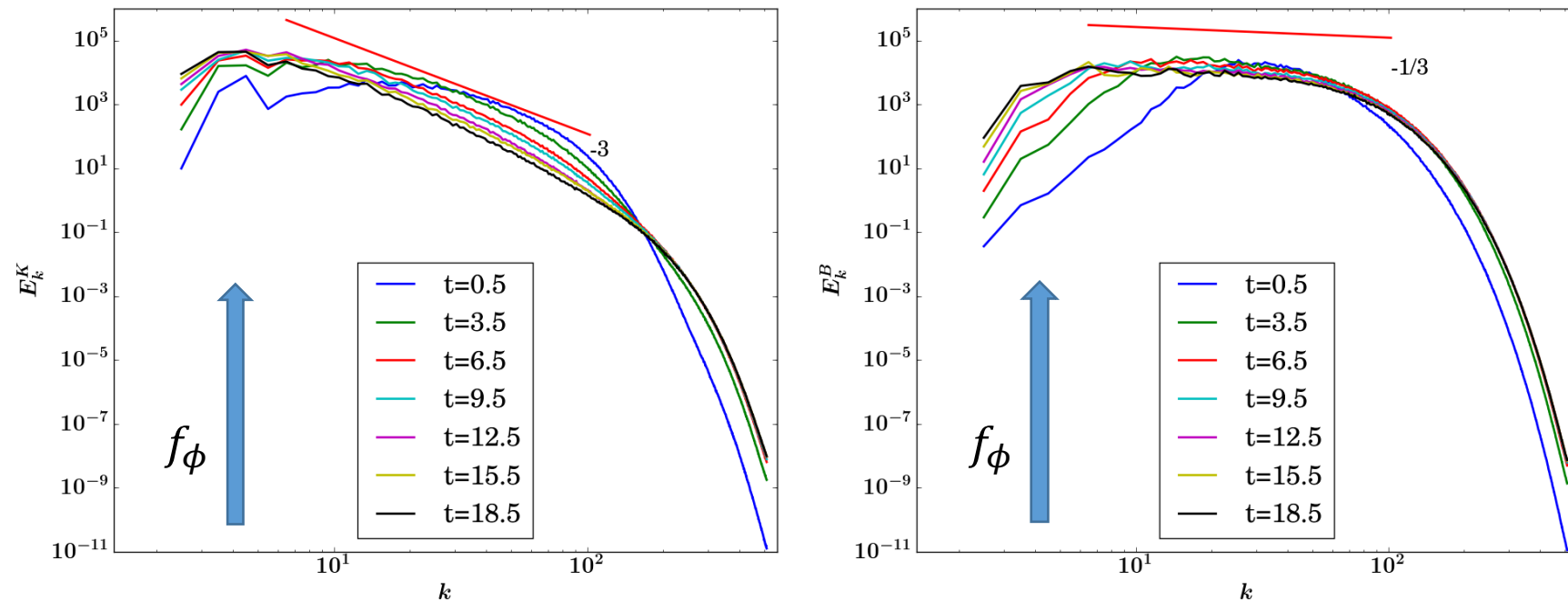
$$H_k^\psi \sim \epsilon_{H\psi}^{2/3} k^{-7/3}$$

H_k^ψ / H_k^A spectrum power law: -7/3

- The −7/3 power is robust. It does not change with the magnitude of external forcing, as long as the separation between the Hinze scale and the dissipation scale is maintained, so the elastic range is long enough.

Energy Spectrum Power Law

- $E_k^K \sim k^{-3}$ in 2D CHNS turbulence; on the other hand, it is well known that $E_k^K \sim k^{-3/2}$ in MHD turbulence.

- The -3 power law is consistent with the direct enstrophy cascade in 2D NS turbulence.

- The -3/2 power law comes from the Alfven effect (Iroshnikov-Kraichnan Theory).

Energy Spectrum Power Law

- $E_k^K \sim k^{-3}$ for 2D CHNS; $E_k^K \sim k^{-3/2}$ for 2D MHD. Why? Why not -3/2?

- This initially surprising result is plausible because in the 2D CHNS system, B_ψ vanishes in most regions. Back reaction is apparently limited.

- On the other hand, the magnetic fields in MHD are not localized at specific regions, and Alfven waves can be everywhere.

- Define the interface packing fraction P to be the ratio of mesh grid number where $|\vec{B}_\psi| > B_\psi^{rms}$ (or $|\vec{B}| > B^{rms}$) to the total mesh grid number.

MHD
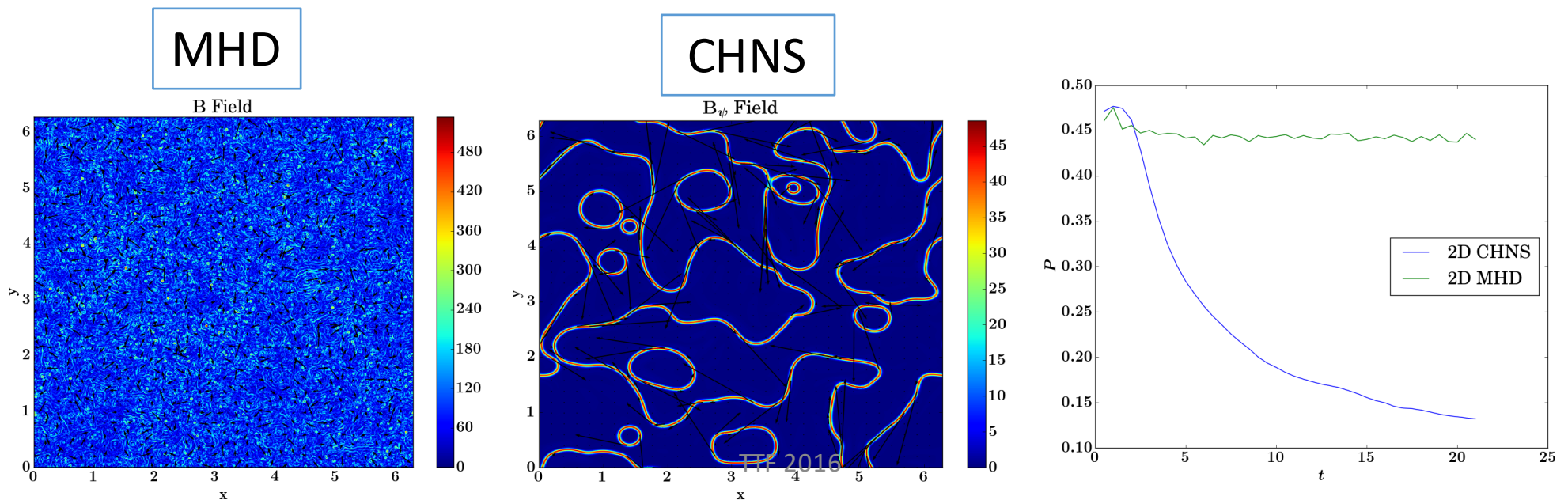
CHNS

Conclusion & Discussion

- ## Comparison:

	2D MHD	2D CHNS
Ideal Quadratic Conserved Quantities	Conservation of E, H^A and H^C	Conservation of E, H^ψ and H^C
Role of elastic waves	Alfven wave couples $\mathbf{v}$ with $\mathbf{B}$	CHNS linear elastic wave couples $\mathbf{v}$ with $\mathbf{B}_\psi$
Origin of elasticity	Magnetic field induces elasticity	Surface tension induces elasticity
Origin of the inverse cascades	The coalescence of magnetic flux blobs	The coalescence of blobs of the same species
The inverse cascades	Inverse cascade of H^A	Inverse cascade of H^ψ
Power law of spectra	$H_k^A \sim k^{-7/3}$	$H_k^\psi \sim k^{-7/3}$

- ## Contrast:

	2D MHD	2D CHNS
Diffusion	A simple positive diffusion term	A negative, a self nonlinear, and a hyper-diffusion term
Range of potential	No restriction for range of A	$\psi \in [-1, 1]$
Kinetic energy spectrum	$E_k^K \sim k^{-3/2}$	$E_k^K \sim k^{-3}$
Suggestive cascade by E_k^K	Suggestive of direct energy cascade	Suggestive of direct enstrophy cascade
Back reaction	$\mathbf{j} \times \mathbf{B}$ force can be significant	Back reaction is apparently limited

Conclusion & Discussion

- The study of elastic turbulence gives us useful lessons.

- The H_k^{ψ} spectrum (inverse cascade of H^{ψ}) is robust. It is an analogue to H_k^A spectrum (inverse cascade of H^A) in 2D MHD. On the other hand, the energy spectrum is NOT so robust. This implies we should perhaps pay more attention on other spectra than the energy spectrum.

- The spatial representations - i.e. tracking contours of bubble surfaces - may be more revealing way of representing the turbulence the traditional power law spectrum.