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Effect of superbanana diffusion on fusion reactivity in stellarators

Fred L. Hinton[a)]
Center for Astrophysics and Space Sciences, University of California, San Diego, La Jolla, California 92093-0424, USA

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Fusion reactivity is usually obtained using a Maxwellian distribution. However, energy-dependent radial diffusion can modify the energy distribution. Superbanana diffusion is energy-dependent and occurs in nonaxisymmetric magnetic confinement devices, such as stellarators, because of ripple-trapped particles which can take large steps between collisions. In this paper, the D-T fusion reactivity is calculated using a non-Maxwellian energy distribution obtained by solving the Fokker-Planck equation numerically, including radial superbanana diffusion as well as energy scattering. The ions in the tail of the distribution, with energies larger than thermal, which are most needed for fusion, are depleted by superbanana diffusion. In this paper, it is shown that the D-T fusion reactivity is reduced by tail ion depletion due to superbanana diffusion, by roughly a factor of 0.5 for the parameters used in the calculation. © *2012 American Institute of Physics*. [http://dx.doi.org/10.1063/1.4748138]

I. INTRODUCTION

The fusion power output of a fusion reactor[1] depends on the fusion reactivity for the fuel species used. Fusion reactivity is defined here as the average over minor radius, of the integral of the energy-dependent fusion rate for the fusion reaction being considered, multiplied by the energy distribution function for the fuel ions. It is usually calculated using a Maxwellian distribution. However, energy-dependent radial diffusion can modify the energy distribution. Superbanana diffusion is energy-dependent and occurs in nonaxisymmetric magnetic confinement devices, such as stellarators, because of ripple-trapped particles which can take large steps between collisions. This tends to deplete the ions in the tail of the distribution function, with energies larger than thermal, which are the ones most needed for fusion.

There are two important regimes of collisionality for superbanana diffusion. In the "$1/\nu$ regime," the single-ion diffusion coefficient is proportional to $1/\nu$ and increases with ion energy E as $E^{7/2}$. Since the rate of repopulation of high-energy ions by Coulomb scattering decreases with energy as $E^{-3/2}$, the tail of the energy distribution for which the fusion reaction rate is appreciable would be strongly depleted. However, at lower collisionality, which occurs at higher particle energy, the appropriate collisionality regime is the "ν regime," where the diffusion coefficient is proportional to ν. Because this decreases with energy as $E^{-3/2}$, it implies that the tail would not be strongly depleted. Also, the $1/\nu$ regime diffusion coefficient depends on the magnetic field structure through the effective helical field ripple, which increases from the plasma center to the edge, while the ion density and temperature decrease toward the edge. This implies that superbanana diffusion would not have as strong an effect on the radially averaged fusion reactivity as indicated above. Clearly, a quantitative assessment of the effect of superbanana diffusion, including radial profile effects, is needed.

The ion distribution function has been obtained by solving the Fokker-Planck equation numerically, including radial superbanana diffusion as well as energy scattering. A steady-state solution has been found in which these two diffusion processes are balanced. The D-T fusion reactivity has been calculated using this distribution function, which gives a quantitative assessment of the effect of superbanana diffusion on fusion reactivity in stellarators.

The standard neoclassical result for the superbanana diffusion coefficient D_{sb} was given by Galeev and Sagdeev.[2] It was expressed in terms of the helical field ripple ϵ_h for a simple model stellarator magnetic field. There are two important regimes of collisionality,[3] or equivalently ion energy: the $1/\nu$ regime for tail ions with energies somewhat higher than thermal, and the ν regime for tail ions with still higher energies, whose collision rates are smaller.

In the $1/\nu$ regime, Nemov *et al.* showed[4] that D_{sb} is given by the standard neoclassical expression with the helical field ripple ϵ_h replaced by an effective ripple modulation amplitude, ϵ_{eff}. They expressed the effective ripple in terms of a weighted integral of the geodesic curvature along the magnetic field line covering the magnetic surface, which can be evaluated numerically, for a given magnetic field geometry. This would include the effect of the diamagnetic magnetic well produced by the plasma at higher beta (ratio of plasma pressure to magnetic pressure). In particular, for quasi-helically symmetric (QHS) configurations,[5] they showed that $\epsilon_{eff} = 0$, and thus D_{sb} would be zero in the $1/\nu$ regime.

However, exact QHS equilibria do not exist,[6] and so the effective helical ripple cannot be zero for a nonaxisymmetric magnetic confinement device. For typical values of ϵ_{eff}, it is shown in this paper that the fusion reactivity is reduced by superbanana diffusion. The effect is not as drastic as indicated by simple considerations taking into account only the $1/\nu$ regime, because of the existence of the ν regime and radial profile effects.

In Sec. II, a simplified Fokker-Planck equation for the tail ions is given, with superbanana diffusion included along

a)Author to whom correspondence should be addressed. Electronic mail: flhinton1@yahoo.com.

with energy scattering. In Sec. III, superbanana diffusion is discussed, and a connection formula for the diffusion coefficient is given to include both collisionality regimes. In Sec. IV, the Fokker-Planck equation for the tail particles is solved numerically to obtain the energy dependence of the tail ion distribution function, for different values of the parameter $\delta = \epsilon_{eff}(1)$, the effective helical field ripple at the edge of the plasma. In Sec. V, this distribution is used to calculate the fusion reactivity as a function of the parameter δ. A summary and conclusions are given in Sec. VI.

II. SIMPLIFIED FOKKER-PLANCK EQUATION FOR TAIL IONS

Tail ions are defined as those with speeds somewhat greater than the ion thermal speed. Their density is assumed to be much less than that of the thermal ions, and they are treated as test particles,[7] using the linearized ion-ion collision term. The thermal ions are assumed to have a Maxwellian distribution. Collisions with electrons may be neglected, for the ion energies of interest. Although a 50-50 D-T mixture of fuel ions is of interest here, a single-mass model is used with an effective mass $m_i = 2.5\, m_p$, where m_p is the proton mass.

The radial transport process is assumed to be dominated by superbanana diffusion. The derivation of the superbanana diffusion coefficient[2] assumes that the bounce frequency of the helical ripple-trapped ions is larger than any other frequency. This leads to a bounce-averaged kinetic equation in which the averaged poloidal precession and radial drift appear, along with pitch-angle scattering. For ions with energies higher than thermal, energy scattering can be considered weaker than pitch-angle scattering and appears together with radial diffusion in the next order equation. This equation, obtained by averaging over pitch angles and a magnetic surface, can be written as follows, using the dimensionless time $\tilde{t} = \nu_{00} t$, where

$$\nu_{00} = 8\pi n_0 e^4 \ln\Lambda/(m_i^2 v_{i0}^3) \tag{1}$$

with n_0 and v_{i0} the ion density and thermal speed at $r = 0$, respectively; the dimensionless radius $\rho = r/a$, where a is the plasma radius; and the dimensionless kinetic energy $\mathcal{E} = m_i v^2/(2T_{i0})$, where T_{i0} is the ion temperature at $\rho = 0$:

$$\frac{\partial f}{\partial \tilde{t}} = \frac{\tilde{n}_i(\rho)}{\mathcal{E}^{1/2}} \frac{\partial}{\partial \mathcal{E}}\left[H(y^{1/2})\left(f + \tilde{T}_i(\rho)\frac{\partial f}{\partial \mathcal{E}}\right)\right] + \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \tilde{D}_{sb} \frac{\partial f}{\partial \rho}\right). \tag{2}$$

Here, $\tilde{n}_i = n_i/n_0$, $\tilde{T}_i = T_i/T_{i0}$, $H(x) = \Phi(x) - x\Phi'(x)$ with $\Phi(x)$ the error function and $y = \mathcal{E}/\tilde{T}_i$, and $\tilde{D}_{sb} = D_{sb}/\nu_{00} a^2$ is the normalized superbanana diffusion coefficient. Averages over pitch angles and a magnetic surface are not indicated explicitly, and a simplified expression for D_{sb} will be used.

The collision processes included in Eq. (2) are energy diffusion, which generally increases the ion's energy, and friction, which decreases it. Ions diffuse up in energy into the tail, where they are lost radially by superbanana diffusion. A source necessary for a steady state has been assumed to be zero for the tail ions. Heating of the tail ions due to fusion alpha-particle slowing down has been neglected, as well as ion loss due to fusion reactions and direct orbit loss.

III. SUPERBANANA DIFFUSION

In this paper, simple random walk formulas for the superbanana diffusion coefficient D_{sb} are used, depending on the collisionality regime.[3] The most relevant collisionality regime is that for the least energetic part of the tail, since the ions diffusing up in energy from the thermal distribution enter this regime first. This is the most collisional part of the tail, and we assume that this is the $1/\nu$ regime. At higher energies, or lower collision rates, the ν regime applies.

In the $1/\nu$ regime, a simple random walk formula for the superbanana diffusion coefficient including the modification of Nemov *et al.*[4] is

$$D_{sb}^{1/\nu} = (2\epsilon_{eff})^{1/2}(\nu/2\epsilon_{eff})\left(\frac{v_d}{\nu/2\epsilon_{eff}}\right)^2, \tag{3}$$

where ϵ_{eff} is the effective helical field ripple,

$$\nu \equiv \nu_{00}\tilde{n}_i/\mathcal{E}^{3/2} \tag{4}$$

is the pitch-angle scattering frequency, $v_d \simeq (v^2/2)/\Omega_i R$ is the radial guiding center drift velocity, with $\Omega_i = eB/m_i$ the ion gyrofrequency, and R is the major radius. This formula is an estimate for a random walk which involves the fraction $(2\epsilon_{eff})^{1/2}$ of the ions trapped in helical magnetic wells, a step rate $\nu/(2\epsilon_{eff})$ which is their detrapping rate, the 90° collision frequency divided by the well depth. The superbanana orbits are assumed to be interrupted by collisions before they are completed, so the radial step size is $v_d/(\nu/2\epsilon_{eff})$ which is the distance the ions drift before they are detrapped by collisions. The diffusion coefficient thus has a strong energy dependence is an increasing function of the effective helical field ripple and is a decreasing function of the ion density:

$$D_{sb}^{1/\nu} \propto \epsilon_{eff}^{3/2}\, \mathcal{E}^{7/2}/\tilde{n}_i. \tag{5}$$

The ν regime occurs at lower collision frequencies, where $\nu/(2\epsilon_{eff}) < \Omega_\theta$, with Ω_θ the poloidal precession frequency. A random walk formula is obtained by assuming that superbanana orbits are infrequently interrupted by collisions, so the step size is v_d/Ω_θ:

$$D_{sb}^{\nu} = (2\epsilon_{eff})^{1/2}(\nu/2\epsilon_{eff})\left(\frac{v_d}{\Omega_\theta}\right)^2. \tag{6}$$

The poloidal precession frequency is $\Omega_\theta = \Omega_{\theta B} + \Omega_{\theta E}$, where $\Omega_{\theta B}$ is due to the poloidal grad-B drift and $\Omega_{\theta E}$ is due to the poloidal $E \times B$ drift. The first of these is[8]

$$\Omega_{\theta B} = -\frac{v^2}{\Omega_i a^2}\epsilon_{h0} \tag{7}$$

(using a helical ripple profile $\epsilon_h(\rho) = \epsilon_{h0}\rho^2$). The second of these can be obtained by assuming an electrostatic potential profile $\phi = \phi_0 \rho^2$, where $\phi_0 > 0$ (assuming the "ion root" for the ambipolar potential), and is

$$\Omega_{\theta E} = -\frac{v_{i0}^2}{\Omega_i a^2} \tilde{T}_i. \tag{8}$$

A connection formula was used in the calculation to cover the entire range of interest for ion energy, effective helical ripple, and ion density. Since the condition for the ν regime is equivalent to the condition that $D_{sb}^{\nu} < D_{sb}^{1/\nu}$, the following connection formula was used:

$$D_{sb} = \left[\frac{1}{D_{sb}^{\nu}} + \frac{1}{D_{sb}^{1/\nu}}\right]^{-1}, \tag{9}$$

which approximately picks out the smaller of the two expressions.

The thermal ion density and temperature profiles were assumed to be given by

$$\tilde{n}_i = \frac{(1-\rho_0)^2 - 0.9(\rho-\rho_0)^2}{(1-\rho_0)^2},$$
$$\tilde{T}_i = \frac{(1-\rho_0)^2 - 0.7(\rho-\rho_0)^2}{(1-\rho_0)^2}, \tag{10}$$

where $\rho_0 = 0.01$ (nonzero to avoid the singularity in Eq. (2) at $\rho = 0$). The radial profile of the effective helical ripple was assumed to be given by

$$\tilde{\epsilon}_{eff}(\rho) = 0.1 + 0.9\rho^4, \tag{11}$$

which is strongly increasing toward the plasma edge, similar to some calculated profiles.[9,10]

As shown in Fig. 1, the superbanana diffusion coefficient increases strongly with energy $\mathcal{E}$ for small minor radius ρ, but not for large minor radius. This is because the $1/\nu$ regime occurs mainly for small ρ, for the chosen parameters.

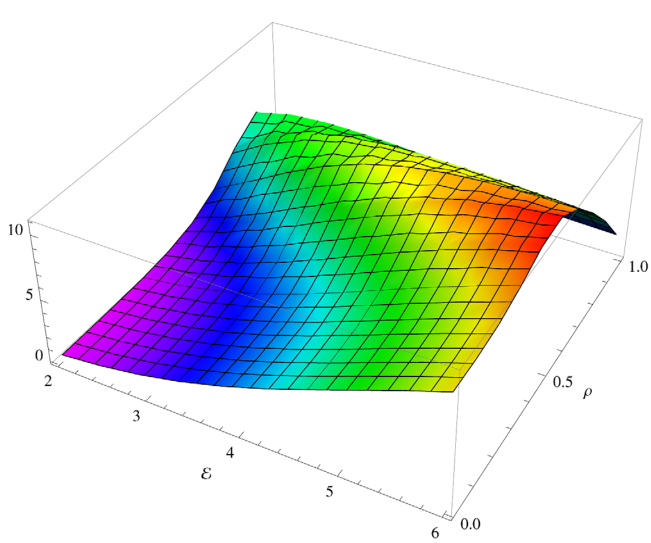

FIG. 1. Normalized superbanana diffusion coefficient as a function of the dimensionless energy $\mathcal{E}$ and radius ρ.

IV. NUMERICAL SOLUTION FOR THE STEADY-STATE TAIL ENERGY DISTRIBUTION

Only the steady-state solution of Eq. (2) is of interest. It was obtained by solving an initial-value problem and using the solution after it is no longer changing with time. It was solved in an energy interval $\mathcal{E}_1 \le \mathcal{E} \le \mathcal{E}_2$; the energy boundaries used in the calculation were $\mathcal{E}_1 = 2.0$ and $\mathcal{E}_2 = 6.0$. The condition $\mathcal{E}_2 \ll (m_i/m_e)^{1/3}$, for justifying the neglect of collisions with electrons, is reasonably well satisfied.

The initial condition was taken to be a Maxwellian with its temperature equal to that of the thermal distribution:

$$f(\mathcal{E}, \rho, 0) = n_f(\rho) \exp[-(\mathcal{E} - \mathcal{E}_1)/\tilde{T}_i(\rho)], \tag{12}$$

where the initial fast-ion density was chosen as

$$n_f(\rho) = 1 - (\rho - \rho_0)^2/(1-\rho_0)^2. \tag{13}$$

The boundary condition at the low energy boundary $\mathcal{E} = \mathcal{E}_1$ was taken to be

$$f(\mathcal{E}_1, \rho, t) = n_f(\rho) \tag{14}$$

consistent with the initial condition. The boundary condition at the high energy boundary $\mathcal{E} = \mathcal{E}_2$ was taken to be

$$f(\mathcal{E}_2, \rho, t) + \tilde{T}_i(\rho)(\partial f/\partial \mathcal{E})_{\mathcal{E}\to\mathcal{E}_2} = 0, \tag{15}$$

which means the ion flux to higher energies is zero. The equation was solved in the radial range $\rho_0 \le \rho \le 1$ with the boundary conditions

$$(\partial f/\partial \rho)_{\rho\to\rho_0} = 0, \quad f(\mathcal{E}, 1, t) = 0. \tag{16}$$

The fast ion density goes to zero at the plasma edge, even though the thermal ion density does not.

The machine parameters used in the calculation are central ion density $n_{i0} = 10^{20}\, m^{-3}$, magnetic field (taken to be constant) $B = 5.0\, T$, major radius $R = 10.0\, m$, minor radius $a = 1.0\, m$, and helical ripple at the plasma edge $\epsilon_{h0} = 0.1$. Calculations were carried out for central ion temperatures of $T_{i0} = 8$ keV, $T_{i0} = 10$ keV, and $T_{i0} = 12$ keV. The ratios of the radially averaged fusion reactivity to the value for a Maxwellian were calculated for values of the parameter $\delta = \epsilon_{eff}(1)$ from 2.5×10^{-4} to 3.0×10^{-2}.

A check on the convergence in time of the solution is provided by the steady-state flux conservation condition. By omitting the time derivative in Eq. (2), multiplying it by $\rho\,\mathcal{E}^{1/2}$, and integrating over ρ from ρ_0 to 1 and over $\mathcal{E}$ from $\mathcal{E}_1$ to $\mathcal{E}_2$, using the boundary conditions, we obtain

$$\int_{\rho_0}^{1} \rho\, d\rho\, \tilde{n}_i(\rho) \left[H(y^{1/2})\left(f + \tilde{T}_i(\rho)\frac{\partial f}{\partial \mathcal{E}}\right)\right]_{\mathcal{E}=\mathcal{E}_1}, \tag{17}$$

$$= \int_{\mathcal{E}_1}^{\mathcal{E}_2} \mathcal{E}^{1/2} d\mathcal{E} \left[D_{sb}(\mathcal{E}, \rho)\frac{\partial f}{\partial \rho}\right]_{\rho=1}. \tag{18}$$

The meaning of this equation is as follows: In a steady state, the ion flux coming into the tail region at the lower energy

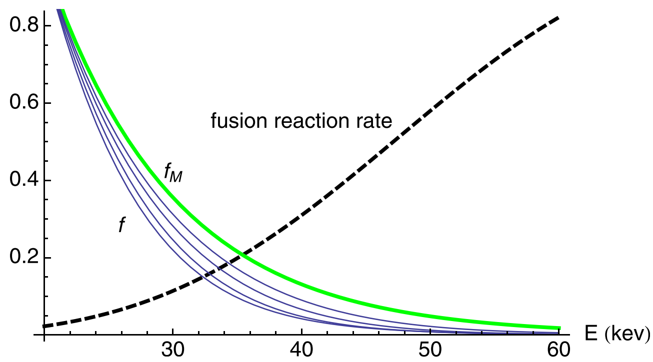

FIG. 2. Computed energy distributions as functions of energy E in keV, along with the Maxwellian and the D-T fusion reaction rate $E\,\sigma(E)$, at $\rho = 0.2$, for $T_{i0} = 10\,\text{keV}$ and for various values of δ (increasing away from the Maxwellian) from 2.5×10^{-3} to 2.0×10^{-2}.

boundary $\mathcal{E} = \mathcal{E}_1$ equals the ion flux leaving at the plasma edge, $\rho = 1$. This condition was found to be reasonably well satisfied at a time $\tilde{t} = 10.0$ and the steady-state distribution function was evaluated at that time.

The computed energy distributions as functions of energy E in keV at $\rho = 0.2$, for various values of δ and the Maxwellian, are shown in Figure 2. Also shown is the D-T fusion reaction rate $E\,\sigma(E)$, defined in Sec. V. This figure shows how the tail ions which are needed for fusion are increasingly depleted, for increasing δ.

The computed solutions for f for $T_{i0} = 10\,\text{keV}$ at $\rho = 0.2$ and at $\rho = 0.8$, for $\delta = 0.01$, and the Maxwellians at $\rho = 0.2$ and at $\rho = 0.8$ are shown in Figure 3. This figure shows that the density of ions which contribute most to the fusion reactivity is reduced by superbanana diffusion for small and larger ρ, when compared with the Maxwellians.

V. FUSION REACTIVITY

The D-T fusion cross section fit given by Bosch and Hale[11] was used. This has the form

$$\sigma(E) = \frac{S(E)}{E}\exp(-B_G/E^{1/2}) \tag{19}$$

in units of $10^{-27}\,\text{cm}^2$, where E is the energy in the center-of-mass frame, in keV. Here, S is a slowly varying function of

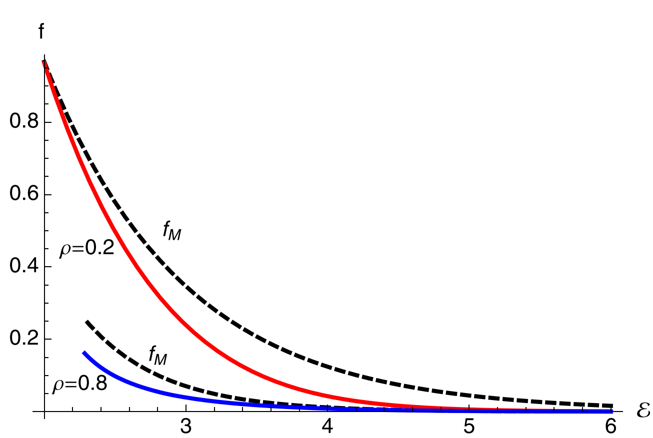

FIG. 3. Computed solutions at $\rho = 0.2$ and at $\rho = 0.8$, for $\delta = 0.01$, and the Maxwellians (corresponding to $\delta = 0.0$).

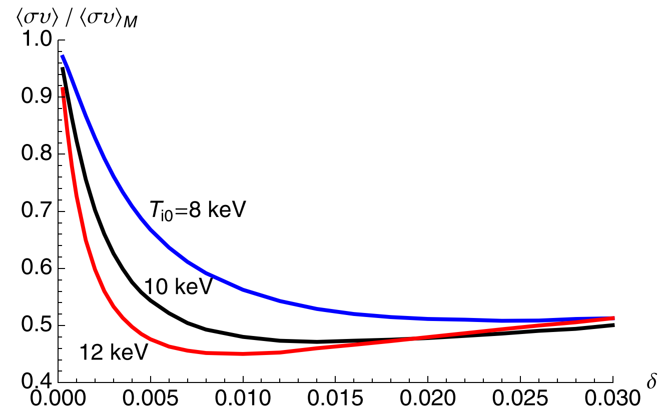

FIG. 4. Radially averaged fusion reactivity divided by the radially averaged value for a Maxwellian, $\langle\sigma v\rangle/\langle\sigma v\rangle_M$, as a function of δ, for three different central ion temperatures.

E, with several fit parameters. The exponential is the tunneling probability, with B_G the Gamov constant. The values of the fit parameters and B_G for the D-T reaction as given by Bosch and Hale were used.

The radially local fusion reactivity is given by integrating $\sigma(E)v$ over energy, where v is the speed, multiplied by the computed distribution function $f(E/T_{i0}, \rho)$. After averaging over ρ, we obtain

$$\langle\sigma v\rangle = C\int_0^1 \rho\, d\rho \int_0^\infty dE\, E\, \sigma(E) f(E/T_{i0}, \rho), \tag{20}$$

where C is a constant. The fusion reaction rate mentioned in Sec. IV is defined as the factor $E\,\sigma(E)$ in the integrand. In evaluating this integral, contributions from energies less than $\mathcal{E}_1$ and greater than $\mathcal{E}_2$ were neglected. The fusion reactivity for a Maxwellian is denoted by $\langle\sigma v\rangle_M$. The radially averaged fusion reactivity divided by the radially averaged value for a Maxwellian, $\langle\sigma v\rangle/\langle\sigma v\rangle_M$, as a function of δ, for three different ion temperatures is shown in Figure 4. This figure shows that the fusion reactivity is reduced by superbanana diffusion when compared with a Maxwellian, for the parameters used in the calculation. The reduction is roughly a factor of 0.5 for all three ion temperatures.

VI. SUMMARY AND CONCLUSIONS

The steady-state solution of the Fokker-Planck equation, Eq. (2), was obtained numerically using the initial and boundary conditions given by Eqs. (12)–(16). The steady-state solution was used to calculate the radially averaged D-T fusion reactivity. It was found that the fusion reactivity is reduced by superbanana diffusion when compared with a Maxwellian, by roughly a factor of 0.5 for the parameters used in the calculation, for central ion temperatures $T_{i0} = 8\,\text{keV}, 10\,\text{keV}$, and $12\,\text{keV}$. The effect is not as drastic as indicated by simple arguments based on the $1/\nu$ regime because of the existence of the ν regime and radial profile effects.

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