Ion-acoustic Shocks with Reflected Ions

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Abstract

Analytic solution for ion-acoustic collisionless shocks with reflected ions is constructed. It extends a classic soliton propagating at the Mach numbers $M < M_{*} \approx 1.6$ beyond this value at which the soliton reflects the upstream ions. The soliton turns into a shock whose parameters, such as the maximum and the minimum of the potential in its trailing nonlinear wave, are obtained in terms of the number of reflected ions. The latter can be related to the shock Mach number, which is implemented for a particular, "box" (piece-wise constant) distribution of upstream ions. The reflected ions, propagating at the double shock speed, support a shock precursor (foot) whose potential is also obtained. The process of expansion of reflected ions into the upstream medium is self-similar, resembling that of the gas expansion into vacuum.

Introduction

- understanding of collisionless shocks is important for:
  - particle acceleration in astrophysical settings
  - supernova remnants, pulsar and black hole magnetospheres,
  - laser-based tabletop proton accelerators
  - inertial confinement

  ....
- only laminar shocks of limited Mach numbers are well understood, largely using soliton approach
- when Mach number exceeds a critical value $M = M_*$, a number of phenomena occur, starting from ion reflection off the shock front

Problem setting and summary of results

- the main question here is whether a quasi-laminar shock transition can still be constructed, or else the shock becomes turbulent
- below we show that when $M > M_*$ the stationary shock transition can continue, albeit in a different form
- besides its departure from the solitary form, the shock acquirers a wave train downstream which is strictly periodic in collisionless regime, Fig.1
- the new analytic solution has a higher limiting Mach number $M = M_{**} > M_*$, when all incident ions are reflected and a stagnated ion flow fills the downstream medium

Summary of results cont'd

- the degenerate state of complete reflection can not be extended to higher amplitudes and Mach numbers and the flow is likely to be non-stationary beyond this point
- in the range $M_* < M < M_{**}$, the only time dependent part of the solution concerns the leading edge of the group of reflected ions
- they form a pedestal attached to the shock potential profile upstream and escape with double the shock speed, Fig.1
- their further fate is determined by a relatively slow spreading of the initially sharp front edge
- by even a small velocity dispersion ions with higher initial velocity undergo additional electrostatic acceleration by passing through the shock pedestal

Shock Model

- ion-acoustic soliton solutions first obtained for the Boltzmann electrons [1] and extended later to the case of adiabatically trapped electrons [2]
- ions were assumed cold which strictly limits the maximum Mach number to $M_* = 1.6$ and $M_* = 3.1$ for Boltzmann and adiabatic electrons, respectively
- we seek to extend the soliton solution beyond the ion reflection point $M > M_*$ while adhering to the exact treatment with an explicit form of the nonlinear dispersion relation $M = M(\phi_{\max})$, where $\phi_{\max}$ is the soliton amplitude
- to include the reflected ions adequately, the assumption about cold upstream ions must be abandoned

Incoming Ion distribution

- goal is to obtain a solution that continuously depends on the ion reflection ratio

$$\alpha = n_{\mathrm{refl}}/n_{\infty}$$

- this quantity can always be calculated given the shock speed and the upstream ion distribution
- assume a “box” distribution with the finite thermal velocity $V_{Ti} = v_2 - v_1$:

$$f_i^{\infty}(v) = \frac{1}{v_2 - v_1}\begin{cases} 1, & -v_2 < v < -v_1 \\ 0, & v \notin (-v_2, -v_1) \end{cases}$$

use dimensionless $e\phi/T_e \to \phi$, measure coordinate in units of $\lambda_D = \sqrt{T_e/4\pi e^2 n_{\infty}}$, ion velocity in $C_s = \sqrt{T_e/m_i}$.

Ion Density and Shock Potential

Assume soliton propagates in with a nominal speed $U = \sqrt{2\phi_{\max}}$, where $\phi_{\max} = \phi(0)$ is the maximum of its potential and $v_1 \le U \le v_2$. The ion density upstream and downstream can be written as follows, Fig.2

$$n_i(\phi) = \frac{1}{v_2 - v_1} \begin{cases} \sqrt{v_2^2 - 2\phi} - \sqrt{U^2 - 2\phi}, \\ \sqrt{v_2^2 - 2\phi} + \sqrt{U^2 - 2\phi} - 2\sqrt{v_1^2 - 2\phi}, \\ \sqrt{v_2^2 - 2\phi} + \sqrt{U^2 - 2\phi}, \quad x > 0; \;\; v_1^2/2 \le \phi \le U^2/2 = \phi_{\mathrm{m}} \end{cases}$$

Poisson equation

$$\frac{d^2\phi}{dx^2} = (1+\alpha)\, e^{\phi} - n_i(\phi), \quad where \;\; \alpha = \frac{U - v_1}{v_2 - v_1} \qquad (1)$$

α is the fraction of ions reflected off the shock

Solution

integrate eq.(1) once imposing the condition $\phi'(\phi_{\max}) = 0$

$$\frac{1}{2}\left(\frac{d\phi}{dx}\right)^2 = \Phi(\phi) + \mathscr{F}^{\pm}(\phi) \equiv \Phi^{\pm}(\phi) \qquad (2)$$

where $'+'$ or $'-'$ relate to $x \geq 0$ and $x < 0$. Here

$$\Phi = (1+\alpha)\left(e^{\phi} - e^{U^2/2}\right) + \frac{\left(v_2^2 - 2\phi\right)^{3/2} - \left(v_2^2 - U^2\right)^{3/2}}{3\left(v_2 - v_1\right)} \qquad (3)$$

$$\mathscr{F}^{\pm} = \frac{1}{3\left(v_2 - v_1\right)} \begin{cases} \left(U^2 - 2\phi\right)^{3/2} - 2\left(v_1^2 - 2\phi\right)^{3/2} \vartheta\left(v_1^2 - 2\phi\right), & x \geq 0 \\ -\left(U^2 - 2\phi\right)^{3/2}, & x < 0 \end{cases} \qquad (4)$$

and ϑ is a Heaviside function.

Solution for the main shock

Implicit solution for the potential $\phi(x)$ in the regions $x \gtrless 0$

$$x(\phi) = \pm\frac{1}{\sqrt{2}} \int_{\phi}^{\phi_{\max}} \frac{d\phi'}{\sqrt{\Phi^{\pm}(\phi')}} \tag{5}$$

The condition $d\phi/dx = 0$ at $x = \infty$ that amounts to $\Phi^{+}(\phi = 0) = 0$ yields the dispersion relation for the soliton

$$e^{U^2/2} - 1 = \frac{v_2^3 + U^3 - 2v_1^3 - \left(v_2^2 - U^2\right)^{3/2}}{3(1+\alpha)(v_2 - v_1)} \tag{6}$$

This is a relation between the soliton amplitude $\phi_{\max} = U^2/2$ and its speed (Mach number) $M = (v_1 + v_2)/2$. Assuming $V_{Ti} = v_2 - v_1 \ll U$, we obtain

$$e^{U^2/2} - 1 - U^2 = -\frac{1}{3}(2U)^{3/2}\frac{(1-\alpha)^{3/2}}{1+\alpha} V_{Ti}^{1/2} \tag{7}$$

Shock transition cont'd

Neglecting the r.h.s. (cold ions, $V_{Ti} \to 0$) gives the solution for the critical Mach number $U = M_* \approx 1.6$ [1]. For a finite $V_{Ti} \ll v_{1,2}$ and arbitrary α one obtains

$$U = M_* - \frac{2^{3/2}}{3} \frac{(1-\alpha)^{3/2} V_{Ti}^{1/2}}{(1+\alpha) M_*^{1/2} (M_*^2 - 1)} \tag{8}$$

Downstream ($x < 0$), the potential oscillates between ϕ_{min} and $\phi_{\mathrm{max}} = U^2/2$ given eq.(8). Similarly, one obtains for ϕ_{min}

$$e^{U^2/2} - e^{\phi_{\mathrm{min}}} = \frac{\left(v_2^2 - 2\phi_{\mathrm{min}}\right)^{3/2} - \left(U^2 - 2\phi_{\mathrm{min}}\right)^{3/2} - \left(v_2^2 - U^2\right)^{3/2}}{3(1+\alpha)(v_2 - v_1)} \tag{9}$$

The solution for ϕ_{min} when $\alpha \ll 1$

$$\phi_{\mathrm{min}} \simeq \frac{2M_*^2 \sqrt{\alpha}}{\sqrt{M_*^2 - 1}}$$

Solution for shock transition cont'd

Strong reflection, $1-\alpha \ll 1$ and $V_{Ti} \ll 1-\alpha \ll 1$

$$\phi_{min} \simeq \phi_{max} - \frac{2M_*^2\left(1-\alpha\right)^2}{\left(1+M_*^2\right)^2}$$

For smaller $1-\alpha$

$$\phi_{min} \simeq \phi_{max} - \frac{9}{4}\left(1-\alpha\right) V_{T_i} M_* \left[1 - \sqrt{\frac{V_{Ti}}{2\left(1-\alpha\right)}} \frac{\left(1+M_*^2\right)}{\sqrt{M_*}}\right]^2 \quad (10)$$

This solution cannot be continued to $\alpha = 1$, as $\phi_{\min}$ reaches $\phi_{\max}$ at

$$\alpha = \alpha_c \simeq 1 - V_{Ti}\left(1+M_*^2\right)^2 / 2M_* < 1 \quad (11)$$

Solution near $\alpha = \alpha_c$

Solution with all particles reflected from the top of the shock potential would require a finite density downstream (to neutralize electrons) which would be possible only if the incident ions had zero velocity dispersion (that is why $1 - \alpha_c \sim V_{Ti}$). Therefore, when α decreases to $\alpha = \alpha_c$ a constant solution $\phi(x) \equiv \phi_{\max}$ establishes downstream. We find it by requiring charge neutrality condition fulfilled identically downstream

$$e^{U^2/2} = \frac{1-\alpha}{1+\alpha}\sqrt{\frac{2U}{V_{Ti}(1-\alpha)} + 1},$$

which yields the critical $\alpha = \alpha_c$, eq.(11). Then, the maximum potential $\phi_{\max} = U^2/2$ is determined by

$$U = M_* - \frac{V_{Ti}^2}{3M_*^2}\frac{\left(M_*^2+1\right)^3}{M_*^2-1}$$

Ion Precursor

The leading soliton and the trailing shock in the case $\alpha \neq 0$ propagate through the foot region where the shock-reflected ions elevate the electrostatic potential from zero at infinity to $\psi = \phi_1$ and partially entrain the incident ions. We shift the variable ϕ to ψ: $\psi = \phi + \phi_1$. Denoting the reflected ion density by ρ and neglecting the thermal spread of the incident ions we the Poisson equation rewrites

$$\frac{d^2\psi}{dx^2} = e^{\psi} - \frac{1}{\sqrt{1 - 2\psi/w^2}} - \rho \tag{12}$$

Here the reference frame propagates with the front of reflected ions $w \simeq 2V_{shock}$. By neglecting the thermal dispersion of reflected ions we describe them using hydrodynamic approximation:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}\rho u = 0 \tag{13}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -\frac{\partial \psi}{\partial x} \tag{14}$$

where u is the flow velocity of reflected ions.

Ion Precursor, cont'd

The problem given by eqs.(12-14) has a close relation with the problem of expansion of one gas into the other (or vacuum) [3, 4]. The solution should depend only on the variable $\xi = x/t$. The equations then rewrite

$$(u-\xi)\rho' + \rho u' = 0 \tag{15}$$

$$(u-\xi)u' + Q\rho' = 0 \tag{16}$$

where we have denoted the ξ- derivative by prime and $Q(\rho) = d\psi/d\rho$. The solubility condition with respect to u', ρ':

$$(u-\xi)^2 = \rho Q,$$

or, using eq.(16)

$$\frac{du}{d\psi} = \pm\frac{1}{\sqrt{\rho Q}}, \tag{17}$$

Ion Precursor, cont'd

where $\rho(\psi)$ is given by eq.(12)

$$\rho = e^{\psi} - \frac{1}{\sqrt{1 - 2\psi/w^2}}$$

$Q(\psi) = (d\rho/d\psi)^{-1}$. The solution for u takes the following form

$$u = \int_{\psi}^{\phi_1} d\psi \sqrt{\frac{e^{\psi} - w^{-2}\left(1 - 2\psi/w^2\right)^{-3/2}}{e^{\psi} - \left(1 - 2\psi/w^2\right)^{-1/2}}} \qquad (18)$$

Integration constant ϕ_1 obtained from the requirement that the reflected beam density near the shock comprises an α fraction of that of the incident ions

$$\rho(\xi = -\infty) \equiv \rho_1 = \frac{\alpha}{\sqrt{1 - 2\phi_1/w^2}}$$

Ion Precursor, cont'd

so that for ϕ_1 we obtain

$$e^{\phi_1} = \frac{1+\alpha}{\sqrt{1-2\phi_1/w^2}}$$

which for small α yields

$$\phi_1 \simeq \frac{\alpha}{1-w^{-2}}$$

The reflected beam maximum velocity is reached at $\psi = 0$ in eq.(18). In the case of small α, from eq.(18) we obtain $u_{\max} \simeq 2\sqrt{\phi_1}$ which, of course, follows from the energy conservation of ions passing from the shock foot of height ϕ_1.

Conclusions

- ion acoustic soliton solution limited by $M < M_* \simeq 1.6$ condition is extended beyond this point which also marks the beginning of ion reflection
- new solution may be formally extended to the point where almost all incident ions are reflected off the leading soliton and a constant potential establishes downstream
- dispersive properties of the shock solution that relate its speed, trailing wave train period and the shock amplitude and particle reflection rate, are obtained
- structure of the shock precursor where the reflected ions are accelerated is determined
- the obtained solution has immediate implications for the laser-based tabletop proton accelerators

References

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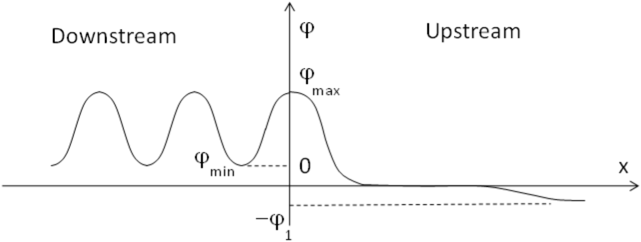

Figure: Electrostatic potential of the shock structure consisting of pedestal, leading soliton and trailing wave

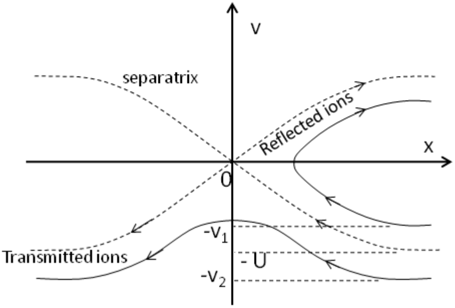

Figure: Phase plane of ions at reflection point.

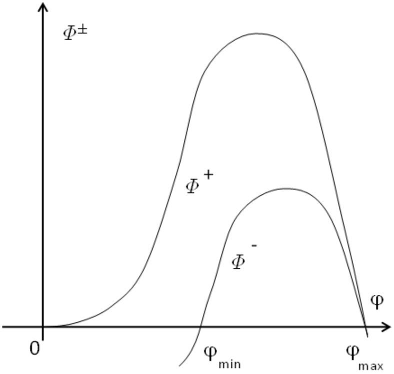

Figure: Pseudo-potentials of "oscillators" described by eq.(2).