Potential Vorticity Dynamics and Models of Zonal Flow Formation

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Outline

- Motivating issues
  - How to represent inhomogeneous PV mixing during relaxation processes?
  - How to calculate spatial PV flux?
- Non-perturbative analyses of the general structure of PV flux
  → structural approach
  - Minimum enstrophy model (selective decay of potential enstrophy)
    → viscous and hyper-viscous transport
  - PV-avalanche model (joint reflection symmetry)
    → K-S equation
- Perturbative analyses of transport coefficients in PV flu
  - Modulational instability, revisited
    → negative viscosity & positive hyper-viscosity
  - Parametric instability
    → Burgers' equation
- Summary

Motivating Issues

- Real space structure of ZF is of practical interest for predictive transport modeling in quasi-2D turbulence. PV mixing in space is essential in ZF generation.

Taylor identity: $$\langle \tilde{v}_y \nabla^2 \tilde{\phi} \rangle = -\partial_y \langle \tilde{v}_y \tilde{v}_x \rangle$$

vorticity flux Reynolds force

- The relaxation dynamics is of fundamental importance in MHD and QG fluid. While Taylor's theory is successful in explaining some plasma experimental results, a relaxation model of vorticity transport is worth researching.

Key points

- turbulence self-organization complex –> What are the general principles?
- PV flux as route to relaxed state –> Is PV homogenization the case??
- zonal flow saturation – how?, especially collisionless cases?

Generic problems

- **How to describe mean PV relaxation to a minimum enstrophy or SOC state?**

PV conservation: $\frac{dq}{dt} = 0$

GFD: Quasi-geostrophic system	Plasma: Hasegawa-Wakatani system
$q = \nabla^2 \phi + \beta y$	$q = n - \nabla^2 \phi$
$\nabla^2 \phi$: relative vorticity; βy: planetary vorticity	n: guiding center (electron density); $\nabla^2 \phi$: polarization (ion density)

Key:

How represent inhomogeneous PV mixing

- Relaxation Principles → General structure of PV flux
- Perturbation theory → Transport coefficients

Non-perturbative analyzes
i) Minimum enstrophy principle

Turbulent magnetic relaxation (J.B. Taylor, 1974)

- minimized magnetic energy subject to constant global magnetic helicity

$$\delta\left[\int d^3x \frac{B^2}{8\pi} + \lambda \int d^3x \vec{A} \cdot \vec{B}\right] = 0$$

$$\Rightarrow \nabla \times \vec{B} = \mu \vec{B}$$

$$\Rightarrow \frac{\vec{J} \cdot \vec{B}}{B^2} = const$$

Taylor state:

- force free B field configuration

- Homogenized $J_{//}$ profile

2D turbulence relaxation (Bretherton & Haidvogel 1976)

- minimized total enstrophy subject to constant total energy

$$\delta\left[\int d^2x \frac{q^2}{2} + \lambda \int d^2x \frac{(\nabla\phi)^2}{2}\right] = 0$$

$$\Rightarrow q = \mu\phi$$

PV stream function

minimum enstrophy state:

- flow structure emergent

	flow	magnetic
conserved quantity (constraint)	total kinetic energy	global magnetic helicity
dissipated quantity (minimized)	fluctuation potential enstrophy	magnetic energy
final state	minimum enstrophy state	Taylor state
structural approach	$\frac{\partial}{\partial t}\Omega < 0 \Rightarrow \Gamma_E \Rightarrow \Gamma_q$	$\frac{\partial}{\partial t}E_M < 0 \Rightarrow \Gamma_H$

- Theory predict end state, but no dynamical insight
  -> flux? what can be said about dynamics?
  -> structural approach (Boozer)

$$\frac{\partial}{\partial t}\int d^3x \frac{B^2}{8\pi} = -\int d^3x \left[\eta J^2 - \Gamma_H \cdot \nabla \frac{\langle J \rangle \cdot B}{B^2} \right] < 0$$

$$\Rightarrow \quad \Gamma_H = -\lambda \nabla \frac{\langle J_\parallel \rangle}{B}$$

PV flux

→ PV conservation

mean field PV: $$\frac{\partial\langle q\rangle}{\partial t}+\partial_y\underbrace{\langle v_y q\rangle}_{\Gamma_q}=\nu_0\partial_y^2\langle q\rangle$$

Γ_q : mean field PV flux

Key Point: what form does PV flux have s/t dissipate enstrophy, conserve energy

selective decay

→ energy conserved $E=\int\frac{(\partial_y\langle\phi\rangle)^2}{2}$

$$\frac{\partial E}{\partial t}=\int\langle\phi\rangle\partial_y\Gamma_q=-\int\partial_y\langle\phi\rangle\Gamma_q \qquad \Rightarrow\Gamma_q=\frac{\partial_y\Gamma_E}{\partial_y\langle\phi\rangle}$$

→ enstrophy minimized $\Omega=\int\frac{\langle q\rangle^2}{2}$

$$\frac{\partial\Omega}{\partial t}=-\int\langle q\rangle\partial_y\Gamma_q=-\int\partial_y\left(\frac{\partial_y\langle q\rangle}{\partial_y\langle\phi\rangle}\right)\Gamma_E$$

$$\frac{\partial\Omega}{\partial t}<0\Rightarrow\Gamma_E=\mu\partial_y\left(\frac{\partial_y\langle q\rangle}{\partial_y\langle\phi\rangle}\right) \qquad \Rightarrow\Gamma_q=\frac{1}{\partial_y\langle\phi\rangle}\partial_y\left[\mu\partial_y\left(\frac{\partial_y\langle q\rangle}{\partial_y\langle\phi\rangle}\right)\right]$$

Structure of PV flux

$$\Gamma_q = \frac{1}{\partial_y \langle \phi \rangle} \partial_y \left[\mu \, \partial_y \left(\frac{\partial_y \langle q \rangle}{\partial_y \langle \phi \rangle} \right) \right] = \frac{1}{\partial_y \langle \phi \rangle} \partial_y \left[\mu \left(- \frac{\langle q \rangle \partial_y \langle q \rangle}{\left(\partial_y \langle \phi \rangle \right)^2} + \frac{\partial_y^2 \langle q \rangle}{\partial_y \langle \phi \rangle} \right) \right]$$

diffusion variable calculated by perturbation theory

diffusion and hyper diffusion of PV

relaxed state:

Homogenization of $\frac{\partial_y \langle q \rangle}{\partial_y \langle \phi \rangle}$

$\neq$

(Prandtl, Batchelor, Rhines, Young)

Homogenization of PV

critical scale $\ell_c \equiv \left(\frac{\partial_y \langle q \rangle}{\partial_y \langle \phi \rangle} \right)^{\frac{-1}{2}}$

$\ell = \ell_c$: Γ_q=0 → ZF growth rate zero

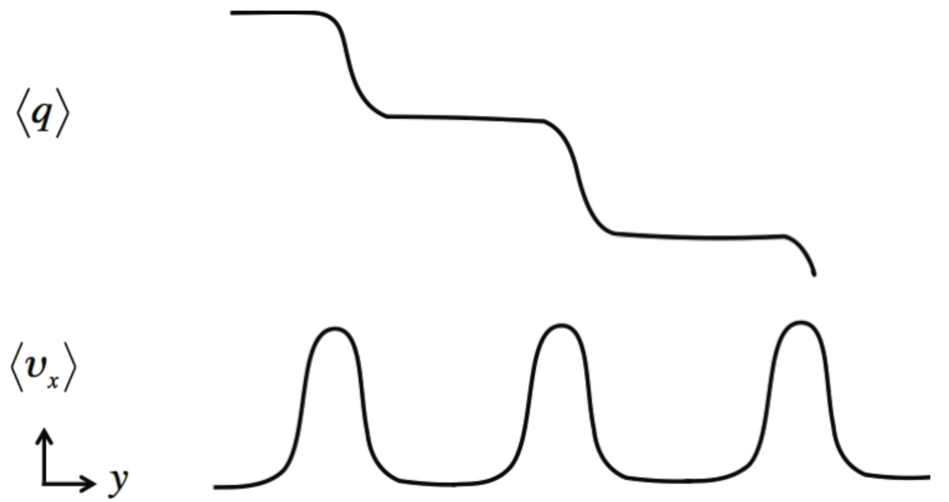
$\ell > \ell_c$: ZF energy growth

$\ell < \ell_c$: ZF energy damping

PV staircase

relaxed state: homogenization of $\dfrac{\partial_y \langle q \rangle}{\langle v_x \rangle}$

→ Zonal flows track the PV gradient → PV staircase



- Highly structured profiles of the staircase is reconciled with the homogenization or mixing process required to produce it.
- Staircase may arise naturally as a consequence of minimum enstrophy relaxation.

The "minimum enstrophy"

- The relaxation rate can be derived by linear perturbation theory about the minimum enstrophy state

$$\langle q \rangle = q_m(y) + \delta q(y,t)$$
$$\langle \phi \rangle = \phi_m(y) + \delta\phi(y,t)$$
$$\partial_y q_m = \lambda \partial_y \phi_m$$
$$\delta q(y,t) = \delta q_0 \exp(-\gamma_{rel} t - i\omega t + iky)$$

$$\gamma_{rel} = \mu\left(\frac{k^4 + 4\lambda k^2 + 3\lambda^2}{\langle v_x\rangle^2} - \frac{8q_m^2(k^2+\lambda)}{\langle v_x\rangle^4}\right)$$
$$\omega = \mu\left(-\frac{4q_m k^3 + 10 q_m k\lambda}{\langle v_x\rangle^3} + \frac{8q_m^3 k}{\langle v_x\rangle^5}\right).$$

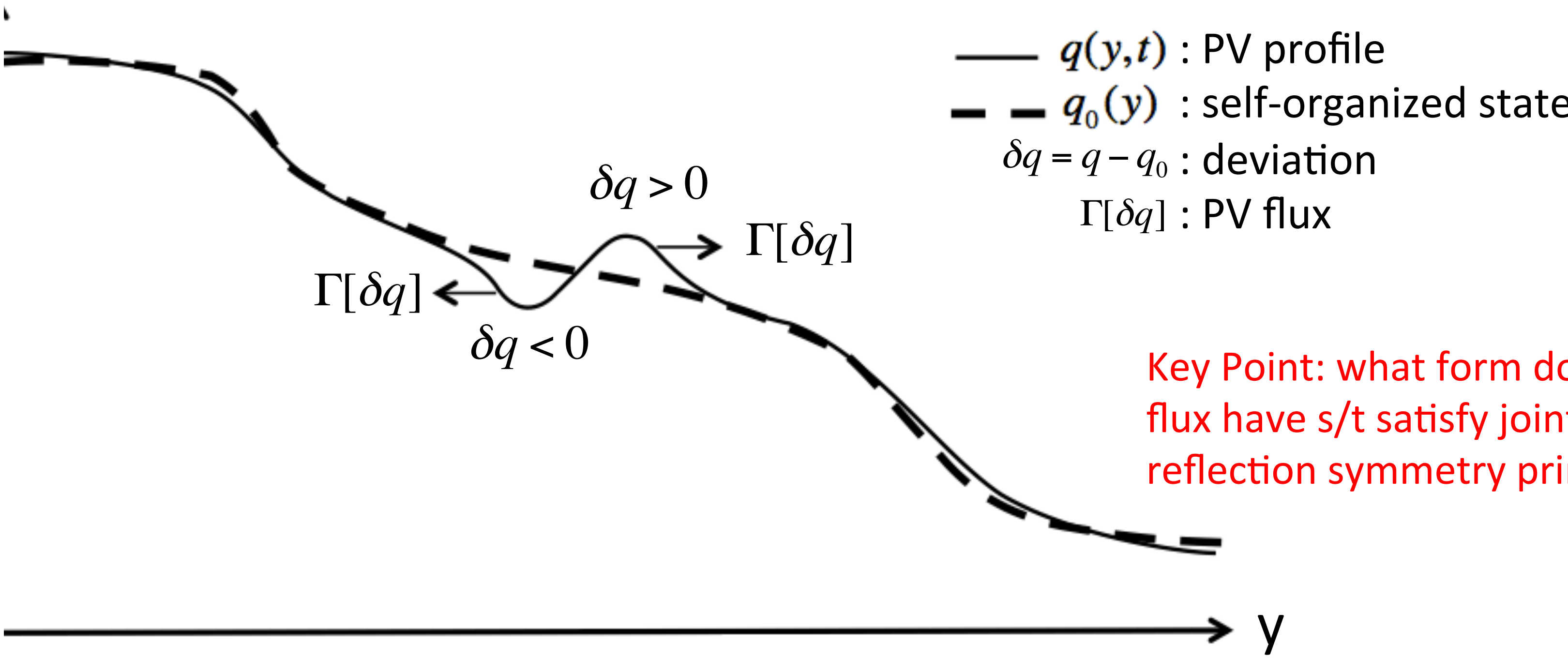
- The condition of relaxation (modes are damped):

$\Upsilon_{rel} > 0$ → $k^2 > \frac{8q_m^2}{\langle v_x\rangle^2} - 3\lambda,$

$k^2 > 0$ → $\frac{8q_m^2}{\langle v_x\rangle^2} > 3\lambda.$ → q_m^2 : the 'minimum enstrophy' of relaxation

- A critical residual enstrophy density is needed in the minimum enstrophy state, so as to sustain a zonal flow of a certain level.

Non-perturbative analyzes
ii) PV-avalanche model



Key Point: what form does PV flux have s/t satisfy joint-reflection symmetry principle

- Joint-reflection symmetry: Γ [δq] invariant under y → -y and δq → -δq

→ $$\Gamma[\delta q] = \sum_l \alpha_l (\delta q)^{2l} + \sum_m \beta_m (\partial_y \delta q)^m + \sum_n \gamma_n \left(\partial_y^3 \delta q\right)^n + \ldots.$$

- large-scale properties : higher-order derivatives neglected
  small deviations : higher-order terms in δq neglected

→ Simplest approximation: $\Gamma[\delta q] = \frac{\alpha}{2}(\delta q)^2 + \beta \partial_y \delta q + \gamma \partial_y^3 \delta q,$

- PV equation: $\partial_t \delta q + \alpha \delta q \partial_y \delta q + \beta \partial_y^2 \delta q + \gamma \partial_y^4 \delta q = 0.$

Kuramoto-Sivashinsky equation

diffusion and hyper diffusion of δq

Non-linear convection of δq

- Avalanche-like transport is triggered by deviation of PV gradient
  → δq implicitly related to the local PV gradient
  → transport coefficients (functions of δq) related to the gradient

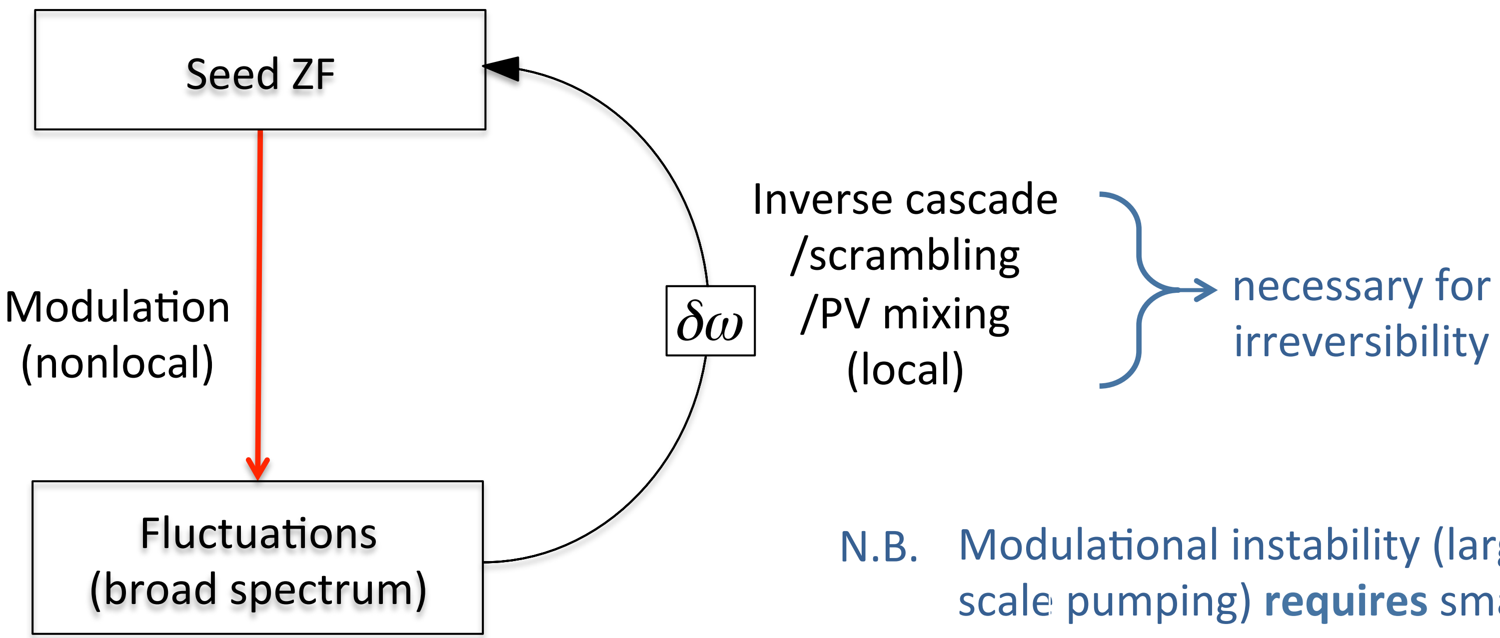
- Convective component of the PV flux can be related to a gradient-dependent effective diffusivity

$$\Gamma_q \sim -D(\partial_y q)\partial_y q \rightarrow -D(\delta q)\delta q$$

$$\Gamma[\delta q] \sim \delta q^2 \rightarrow -D(\delta q)\delta q, \text{ with } D(\delta q) \rightarrow D_0 \delta q$$

Perturbative analyses of PV flux
i) Modulational instability

- The evolution of perturbation (seed ZF) as a way to look at PV transport



N.B. Modulational instability (large scale pumping) **requires** small scale PV mixing <-> irreversibility

Revisiting Modulational Instability

ZF evolution determined by Reynolds force

$$\frac{\partial}{\partial t}\delta V_x = -\frac{\partial}{\partial y}\langle \tilde{v}_x \tilde{v}_y \rangle = \frac{\partial}{\partial y}\sum_{k<}\frac{k_x k_y}{k^4}\tilde{N}_k$$

vorticity flux

$N_k = k^2 |\psi_k|^2 / \omega_k$ is wave action density, for Rossby wave it is proportional to the enstrophy density.

N_k is determined by WKE:

$$\frac{\partial \tilde{N}}{\partial t} + v_g \cdot \nabla \tilde{N} + \delta\omega_k \tilde{N} = \frac{\partial (k_x \delta V_x)}{\partial y}\frac{\partial N_0}{\partial k_y}$$

➜ Turbulent vorticity flux derived

$$\frac{\partial}{\partial t}\delta V_q = -q^2 \delta V_q \sum_k \left(\frac{k_x^2 k_y}{k^4}\right)\frac{\delta\omega_k}{(\omega_q - q\cdot v_g)^2 + \delta\omega_k^2}\frac{\partial N_0}{\partial k_y}$$

κ(q)

q: ZF wavenumber

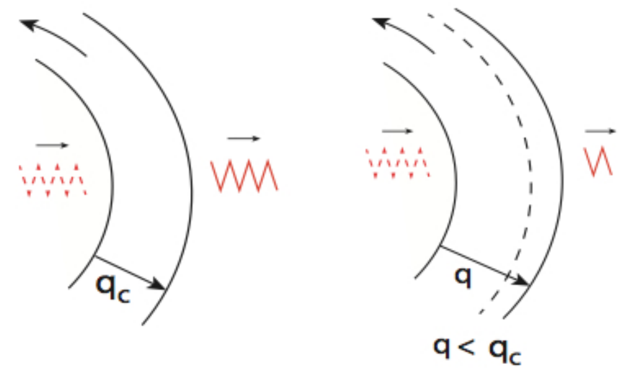
$$\frac{\partial}{\partial t}\delta V_q = -q^2 \kappa(q)\ \delta V_q$$

κ(q) ≠ const at larger *q*

➜ scale dependence of PV flux

➜ non-Fickian turbulent PV flux

- A simple model from which to view *κ (q)*:
  - Defining MFP of wave packets as the critical scale $q_c^{-1} \equiv \upsilon_g \delta\omega_k^{-1}$
  - coarse graining scales smaller than q_c
  - keeping next order term in expansion of response function



$$q^{-1} >> q_c^{-1} \Rightarrow \frac{\delta\omega_k}{(q\upsilon_g)^2 + \delta\omega_k^2} \approx \frac{1}{\delta\omega_k}\left(1 - \frac{q^2}{q_c^2}\right)$$

capture expected and sensible trend!

➜ zonal growth evolution:

$$\partial_t \delta V_x = -q^2 D \delta V_x + q^4 H \delta V_x$$

➜ negative viscosity and positive hyper-viscosity

$$D = \sum_k \frac{k_x^2}{\delta\omega_k k^4} \frac{k_y \partial N_0}{\partial k_y} < 0$$

$$H = -\sum_k q_c^{-2} \frac{k_x^2}{\delta\omega_k k^4} \frac{k_y \partial N_0}{\partial k_y} > 0$$

< 0

(expected wave enstrophy spectrum statistics)

⟺ previous calculation of relaxation models:

$$\frac{\partial \langle \upsilon_x \rangle}{\partial t} = \Gamma_q = \frac{1}{\partial_y \langle \phi \rangle} \partial_y \left[\mu \left(-\frac{\langle q \rangle \partial_y \langle q \rangle}{(\partial_y \langle \phi \rangle)^2} + \frac{\partial_y^2 \langle q \rangle}{\partial_y \langle \phi \rangle} \right) \right]$$

$$\Gamma[\delta q] = \frac{\alpha}{2}(\delta q)^2 + \beta \partial_y \delta q + \gamma \partial_y^3 \delta q,$$

Discussion of *D* and *H*

- Roles of negative viscosity and positive hyper-viscosity **(Real space)**

$$\frac{\partial}{\partial t}\delta V_x = D\partial_y^2\delta V_x - H\partial_y^4\delta V_x$$

$$\frac{\partial}{\partial t}\int\frac{1}{2}\delta V_x^2 d^2x = -D\int\left(\partial_y\delta V_x\right)^2 d^2x - H\int\left(\partial_y^2\delta V_x\right)^2 d^2x$$

$D<0\Rightarrow\gamma_{q,D}>0$	ZF growth (Pumper D)
$H>0\Rightarrow\gamma_{q,H}<0$	ZF suppression (Damper H)

⇒ Energy transferred to large scale ZF

➜ D, H as model of spatial PV/momentum flux beyond over-simplified negative viscosity

- $D = Hq^2$ sets the cut-off scale

$$\Rightarrow l_c^2 = \sqrt{\frac{H}{|D|}}$$

⟺

Minimum enstrophy model

$$\Gamma_q = \frac{1}{\partial_y\langle\phi\rangle}\partial_y\left[\mu\,\partial_y\left(\frac{\partial_y\langle q\rangle}{\partial_y\langle\phi\rangle}\right)\right] \Rightarrow \ell_c \equiv \left(\frac{\partial_y\langle q\rangle}{\partial_y\langle\phi\rangle}\right)^{\frac{-1}{2}}$$

$\ell > \ell_c$: ZF energy growth → D process dominates at large scale

$\ell < \ell_c$: ZF energy damping → H process dominates at small scale

Perturbative analyses
ii) Parametric instability

Pseudo-fluid (wave packets) model

	pseudo-fluid	molecular fluid
distribution function	$N_{\boldsymbol{k}}$	f
scale	$v_g/\delta\omega_k$	mean free path
density	$n^w = \int N_{\boldsymbol{k}} d\boldsymbol{k}$	$n = \int f d\boldsymbol{v}$
momentum	$\boldsymbol{p}^w = \int \boldsymbol{k} N_{\boldsymbol{k}} d\boldsymbol{k}$	$\boldsymbol{p} = \int m\boldsymbol{v} f d\boldsymbol{v}$
energy	$E^w = \int \omega_{\boldsymbol{k}} N_{\boldsymbol{k}} d\boldsymbol{k}$	$E = \int \frac{1}{2} m v^2 f d\boldsymbol{v}$
velocity	$\boldsymbol{V}^w = \frac{\int \boldsymbol{v}_g N_{\boldsymbol{k}} d\boldsymbol{k}}{\int N_{\boldsymbol{k}} d\boldsymbol{k}}$	$\boldsymbol{V} = \frac{\int \boldsymbol{v} f d\boldsymbol{v}}{\int f d\boldsymbol{v}} = \frac{\boldsymbol{p}}{mn}$

mean free path of wave packets

-- pseudo-fluid evolution:

multiplying the WKE by v_{gy} and integrating over k

normalizing by pseudo-density n^{ω}

$$\Rightarrow \frac{\partial}{\partial t}V_y^w + V_y^w \frac{\partial}{\partial y}V_y^w = -a\langle v_x \rangle'$$

inviscid Burgers'eq.
source: zonal shear

$$a = \int \frac{2\beta k_x^2}{k^4}\left(1 - \frac{4k_y^2}{k^2}\right) N_k dk \Big/ \int N_k \, dk$$

-- ZF evolution:

$$\frac{\partial}{\partial t}\langle v_x \rangle = -\frac{\partial}{\partial y} V_y^w P_x^w$$

P_x^{ω} evolves in the same way as $\langle v_x \rangle$

$$\int k_x WKE \; d^2k \Rightarrow \frac{\partial}{\partial t}\int k_x N_k d^2k = -\frac{\partial}{\partial y}\int v_{gy} k_x N_k d^2k$$

- ZF growth rate in monochromatic limit:

  linearizing the above two eqs.

$$\gamma_q = \sqrt{q^2 k_x^2 |\varphi_k|^2 \left(1 - \frac{4k_y^2}{k^2}\right)}$$

The reality of γ_q requires $k_x^2 > 3k_y^2$

$\gamma_q \propto |q|$ indicates convective instability

	PV flux	convective	viscous	hyper-viscous	coefficients
(non-perturb.)	Min. enstrophy relaxation		•	•	
	PV-avalanche relaxation	•	•	•	
(perturbative)	Modulational instability		•	•	$D_t(<0), H_t(>0)$
	Parametric instability	•			$\gamma_q(\sim \lvert q \rvert)$

- Minimum enstrophy $$\frac{\partial \langle v_x \rangle}{\partial t} = \Gamma_q = \frac{1}{\partial_y \langle \phi \rangle} \partial_y \left[\mu \left(-\frac{\langle q \rangle \partial_y \langle q \rangle}{(\partial_y \langle \phi \rangle)^2} + \frac{\partial_y^2 \langle q \rangle}{\partial_y \langle \phi \rangle} \right) \right]$$
- PV-avalanche $$\Gamma[\delta q] = \frac{\alpha}{2}(\delta q)^2 + \beta \partial_y \delta q + \gamma \partial_y^3 \delta q,$$
- Modulational instab. $$\partial_t \delta V_x = -q^2 D \delta V_x + q^4 H \delta V_x$$
- Parametric instab. $$\gamma_q = \sqrt{q^2 k_x^2 |\varphi_k|^2 \left(1 - \frac{4k_y^2}{k^2}\right)}$$

Summary

- Inhomogeneous PV mixing is identified as the fundamental mechanism for zonal flow formation. This study offered new perspectives and approaches to calculating spatial flux of PV. The structure of PV flux is studied by non-perturbative relaxation principles and perturbative analyses of wave kinetic equation. These are synergetic and complementary approaches.
- Vorticity flux from selective decay of enstrophy shows complex structure of diffusion and higher order diffusion terms. The homogenized quantity in the minimum enstrophy state is the ratio of PV gradient to zonal flow velocity. This is consistent with the structure of the PV staircase.
- Vorticity flux from PV-avalanche model is constrained by the joint reflection symmetry condition, and contains diffusive, hyper-diffusive, and convective terms. The convective transport of PV can be generalized to an effective diffusive transport.
- Transport coefficients are derived using perturbation theory. Both relaxation principle and perturbation theory reveal some critical scales at which ZF growth and ZF damping are equal.